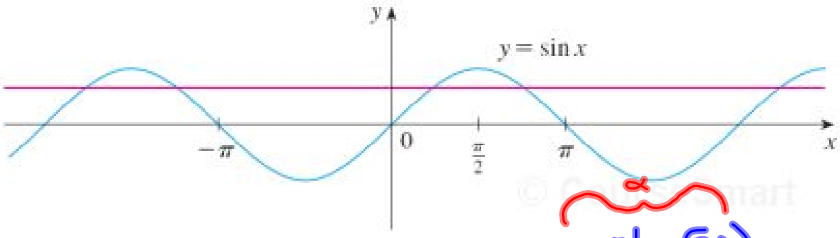
7.6 INVERSE TRIGONOMETRIC FUNCTIONS

$\sin^{-1}(x) = \arcsin(x)$
$\cos^{-1}(x) = \arccos(x)$
$\tan^{-1}(x) = \arctan(x)$.



$\cos(\underbrace{\sin^{-1}(\tfrac{\sqrt{3}}{2})}_{\alpha}) = \frac{1}{2}$

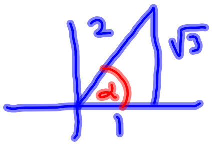
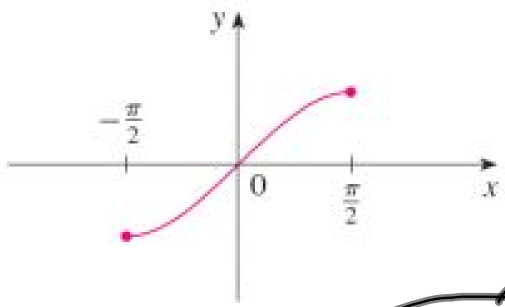


FIGURE 1



Makes sin x 1-to-1

Domain of f

FIGURE 2 $y = \sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

1 $\sin^{-1}x = y \iff \sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Range of f^{-1}

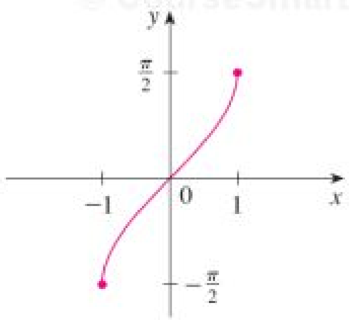


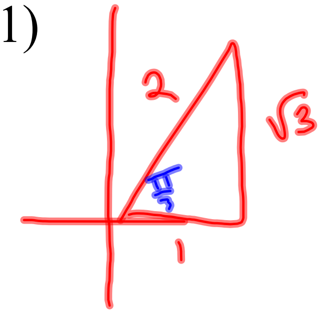
FIGURE 4
$y = \sin^{-1}x = \arcsin x$

1–10 Find the exact value of each expression.

(See also, Example 1)

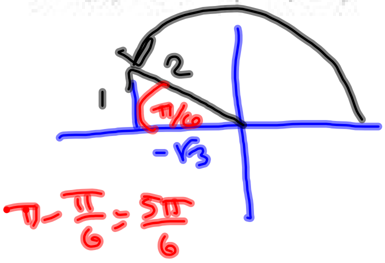
1. (a) $\sin^{-1}(\sqrt{3}/2)$

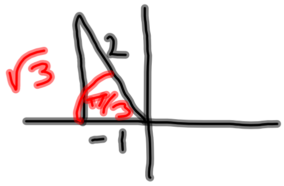
$= \frac{\pi}{3}$



Come Back? Domain Issues.

4. (a) $\cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$ (b) $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$

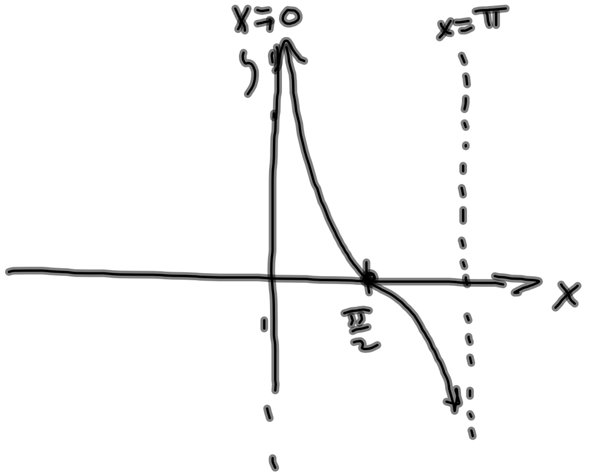




2

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

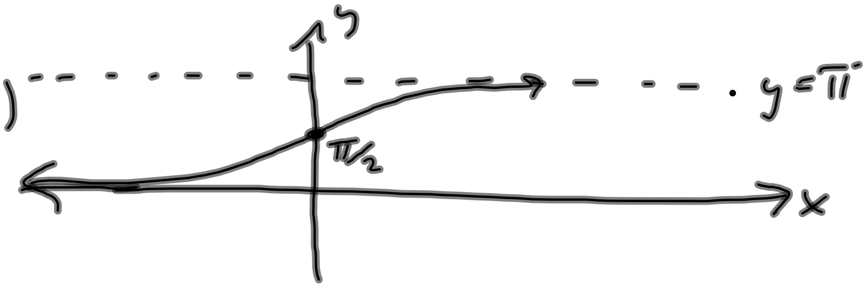
$$\sin(\sin^{-1} x) = x \quad \text{for } -1 \leq x \leq 1$$



We can restrict domain to $(0, \pi)$ to make it 1-to-1.
$\therefore$ $x = \frac{5\pi}{6}$ on last page is AOK.

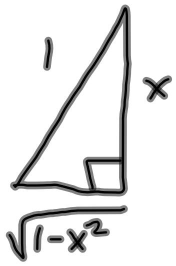
cot(x)

$\cot^{-1}(x)$

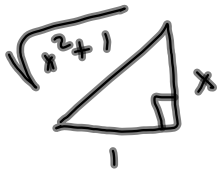


12–14 Simplify the expression.

12. $\tan(\sin^{-1}x)$

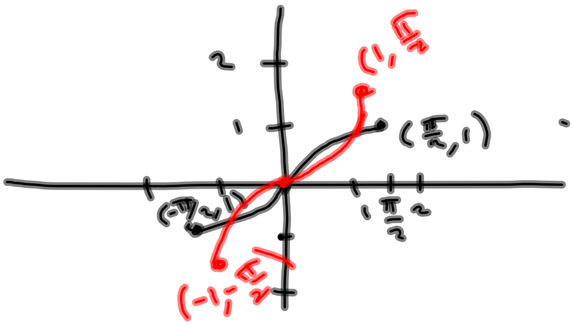


13. $\sin(\tan^{-1}x) = \dfrac{x}{\sqrt{x^2+1}}$



15–16 Graph the given functions on the same screen. How are these graphs related?

15. $y = \sin x,\ -\pi/2 \leqslant x \leqslant \pi/2;\quad y = \sin^{-1}x;\quad y = x$



3 $$\boxed{\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad -1 < x < 1}$$

§7.6 I Thurs.
7.6 II Fri.

Let $y = \sin^{-1}x$. Then $\sin y = x$ and $-\pi/2 \leqslant y \leqslant \pi/2$. Differentiating $\sin y = x$ implicitly with respect to x, we obtain

$$\cos y \frac{dy}{dx} = 1$$

and
$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \geqslant 0$ since $-\pi/2 \leqslant y \leqslant \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Therefore
$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

See Example 2 for an application.

4

$$\cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi$$

5

$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leqslant x \leqslant \pi$$

$$\cos(\cos^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

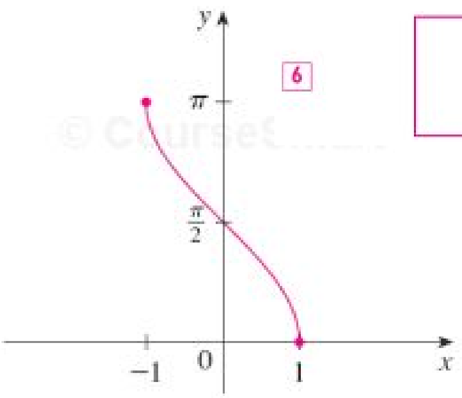


FIGURE 7
$y = \cos^{-1}x = \arccos x$

6

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad -1 < x < 1$$

Aug 31-9:56 AM

7 $$\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

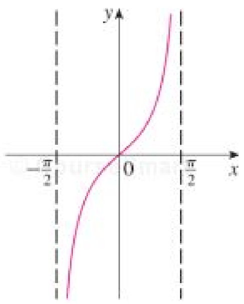


FIGURE 8
$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

8 $$\lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2} \qquad \lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2}$$

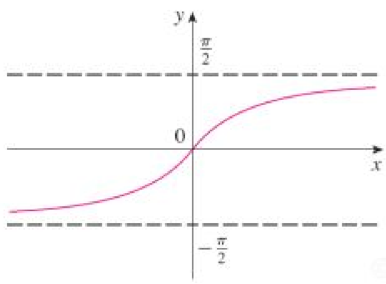


FIGURE 10
$y = \tan^{-1}x = \arctan x$

9 $$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$$

Proof: (Leaving proofs of other derivatives as exercises.)

$y = \tan^{-1}x$. Then $\tan y = x$. Differentiating this latter equation implicitly with respect to x, we have

$$\sec^2 y \frac{dy}{dx} = 1$$

and so $$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

9 $$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$$

Domain chosen for csc (x)

10
$$y = \csc^{-1}x \ (|x| \geq 1) \iff \csc y = x \ \text{ and } \ \boxed{y \in (0, \pi/2] \cup (\pi, 3\pi/2]}$$
$$y = \sec^{-1}x \ (|x| \geq 1) \iff \sec y = x \ \text{ and } \ y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$
$$y = \cot^{-1}x \ (x \in \mathbb{R}) \iff \cot y = x \ \text{ and } \ y \in (0, \pi)$$

#s 19, 20, 21 are the remaining proofs of the derivative formulas for the inverse trig functions. See the discussion on page 458 about choice of domain for csc x and sec x. We'll do a little bit with all of them, but mostly arcsine, arccosine and arctangent.

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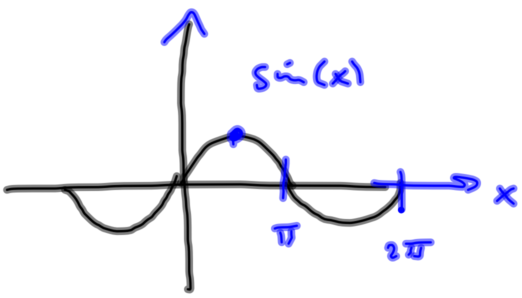
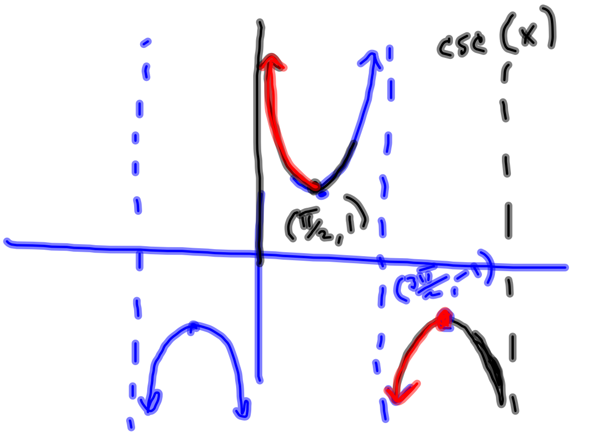




Domain
chosen shown in red.
This will be the range
for $\csc^{-1}(x)$ in the
sequel.

Sep 7-12:13 PM

22–35 Find the derivative of the function. Simplify where possible. $\sqrt{x}^2 = x$ $\sqrt{x^2} = |x|$

23. $y = \tan^{-1}\sqrt{x}$ $= \dfrac{1}{1+x} \cdot \dfrac{1}{2\sqrt{x}}$

Stick these on a cheat sheet. $\dfrac{d}{dx}\left[f(g(x))\right] = \dfrac{df}{dg} \cdot \dfrac{dg}{dx}$

11 TABLE OF DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

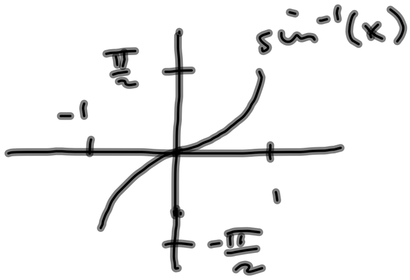
$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

43–46 Find the limit.

43. $\lim_{x \to -1^+} \sin^{-1}x = -\frac{\pi}{2}$



36–37 Find the derivative of the function. Find the domains of the function and its derivative.

36. $f(x) = \arcsin(e^x)$ $= \dfrac{1}{\sqrt{1-e^{2x}}} \cdot e^x$

$(e^x)^2 = e^{2x}$

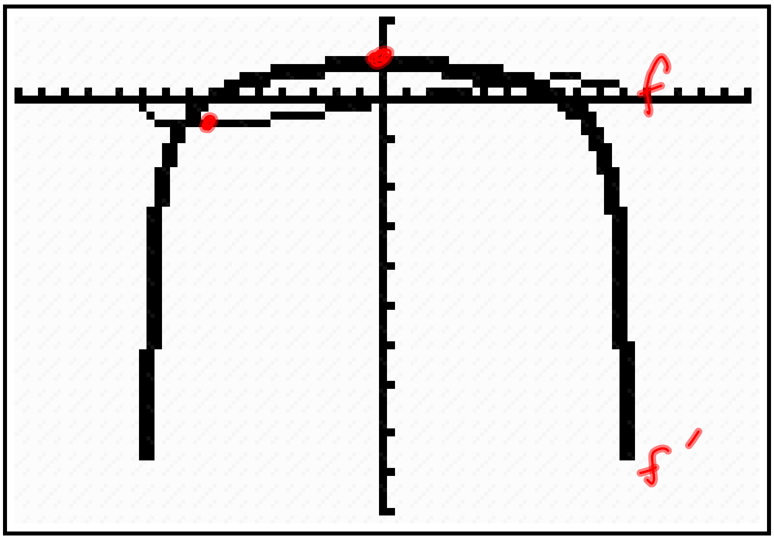
41–42 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

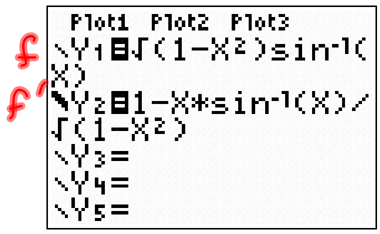
41. $f(x) = \sqrt{1-x^2}\arcsin x$ $= (1-x^2)^{\frac{1}{2}} \arcsin(x)$

$$\Rightarrow f'(x) = \tfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\arcsin(x) + (1-x^2)^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}$$

$$= -\frac{x\arcsin(x)}{\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

$$= \frac{-x\arcsin(x)}{\sqrt{1-x^2}} + 1$$





7.6 I #s 2, 5, 8, 14, 16, 18, 21, 28, 30, 44

7.6 II #s 51, 58, 60, 64, 70

For Thursday, I want these teams of 3 (or 4) to prepare 1 example each.

#48: Beth, Kelly, Silvano, Gabrielle
#50: Ken, Terry, Roy
#54: Matthew, Andy, Ashley

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Sep 6-12:59 PM