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$$\int \frac{\sin(2t)}{1+\cos^2(t)}\,dt = \int \frac{\sin(2t)\,dt}{1+\frac{1+\cos(2t)}{2}}$$

$$= \int \frac{\sin(2t)\,dt}{\frac{3+\cos(2t)}{2}} = 2\int \frac{\sin(2t)}{3+\cos(2t)}\,dt$$

$u = 3+\cos(2t)$

$du = -2\sin(2t)\,dt \longrightarrow dt = -\frac{du}{2\sin(2t)}$

$$= 2\int \frac{\sin(2t)}{3+\cos(2t)} \cdot \frac{du}{-2\sin(2t)} = 2\int \frac{\cancel{\sin(2t)}}{u} \cdot \frac{du}{-\cancel{2\sin(2t)}}$$

$$= -\int \frac{du}{u} = -\ln|u| + c = -\ln|3+\cos(2t)| + c$$

7.5 EXPONENTIAL GROWTH AND DECAY

7.5 #s 1, 4, 6, 8, 11, 13, 18

1 $$\frac{dy}{dt} = ky \Longrightarrow y = e^{kt}$$

when $k=1$, it's just e^t

2 THEOREM The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

Notice that $2^t = \left(e^{\ln(2)}\right)^t = e^{(\ln(2))t} = e^{kt}$

e^{kt} covers pretty much the entire family $y = b^x$. We focus on e^{kt}.

2. A common inhabitant of human intestines is the bacterium *Escherichia coli*. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 60 cells.

(a) Find the relative growth rate. *Punt

(b) Find an expression for the number of cells after t hours.

(c) Find the number of cells after 8 hours.

(d) Find the rate of growth after 8 hours.

(e) When will the population reach 20,000 cells?

k?

[redacted]

$N(t)$ = the number of cells at time t.

~~t is time, in minutes.~~

They gave us doubling time.

$$\cancel{N(20) = 2N(0) = 2N_0}$$

$$\cancel{N_0 e^{20k} = 2N_0}$$

$$\cancel{e^{20k} = 2}$$

$$\cancel{20k = \ln(2)}$$

$$\cancel{k = \frac{\ln(2)}{20}}$$

Exponential Growth

$$N(t) = N_0 e^{kt}$$

$$\cancel{(b)\ \boxed{N(t) = 60e^{\frac{\ln(2)}{20}t}}}$$ oopsie!

Want t to be measured in HOURS!

Doubling time is 20 min $= (20\text{ min})\left(\frac{1\text{ hr}}{60\text{ min}}\right) = \frac{1}{3}$ hr

Don't you dare use .3 or .33 or .333 or .3333

$$N(\tfrac{1}{3}) = 2N_0$$

$$N_0 e^{\frac{1}{3}k} = 2N_0$$

$$e^{\frac{1}{3}k} = 2$$

$$\tfrac{1}{3}k = \ln(2)$$

$$k = 3\ln(2)$$

$\therefore$ $\boxed{N(t) = 60e^{3\ln(2)t} \text{ is the correct response for (b)}}$

(c) $N(8) = 60e^{3\ln(2)(8)}$

$\approx$

(d) Growth rate after 8:

$$\left.\frac{dN}{dt}\right|_{t=8} = \left. 3\ln(2)\cdot 60^{(3\ln(2))t}\right|_{t=8}$$

$$3\ln(2)\cdot 60e^{(3\ln(2))(8)}$$

$\approx$

(e) Solve

$$60e^{kt} = 20000$$

$$e^{kt} = \frac{2000}{6} = \frac{1000}{3}$$

$$kt = \ln(1000/3)$$

$$t = \frac{1}{k}\ln\left(\frac{1000}{3}\right).$$

Sep 2-12:06 PM

Recall
A disk has diameter 5 inches with an error of $\pm .01$ inches.

Relative Error is $\pm \frac{.01}{5}$

$P = $ Pop.

$$\frac{dP}{dt} = kP.$$

Rate of Change is Proportional to Population size.

$$\frac{\frac{dP}{dt}}{P} = k$$

Relative rate of Change is Konstant

This is the k that's in the exponent of e:

$$e^{kt}$$

$$P(t) = P_0 e^{kt}$$

$$P'(t) = \frac{dP}{dt} = kP_0 e^{kt} = \boxed{P_0 e^{kt}} \cdot \frac{d}{dt}[kt]$$

$$\frac{d}{d(kt)}[P(t)] \cdot \frac{d}{dt}[kt]$$

kt is the argument of the function.

$$P(t) = P_0 e^{kt}$$

$$P'(t) = \frac{dP}{dt} = \frac{d}{dt}\left[P_0 e^{kt}\right] = kP_0 e^{kt}$$

Now, k is called relative growthrate, because

$$\frac{\frac{dP}{dt}}{P} = \frac{kP_0 e^{kt}}{P_0 e^{kt}} = k$$

Sep 2-12:30 PM

$$\int \frac{du}{u}$$

$$\int \frac{\sin(2x)}{1+\cos^2 x}\,dx = \int \frac{\sin(2x)}{1+\frac{1+\cos(2x)}{2}}\,dx$$

$$= \int \frac{\sin(2x)}{\frac{3+\cos(2x)}{2}}\,dx = 2\int \frac{\sin(2x)}{3+\cos(2x)}\,dx$$

Sweat it.

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

with respect to

$$\ln(x^3) = 3\ln(x)$$

$$(fg)' = f'g + fg'$$

Sep 1-1:03 PM

5. The table gives estimates of the world population, in millions, from 1750 to 2000:

Year	Population	Year	Population
1750	790	1900	1650
1800	980	1950	2560
1850	1260	2000	6080

(a) Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with the actual figures.

(b) Use the exponential model and the population figures for 1850 and 1900 to predict the world population in 1950. Compare with the actual population.

(c) Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 2000. Compare with the actual population and try to explain the discrepancy.

$P(3) = 102{,}900$

$P(4) = 720{,}300$

1 The population obeys the law of uninhibited growth.

2 The pop's relative growth rate is constant

3 The growth rate is proportional to the population itself.

(a) FIND $P(t)$

(b) When is $P(t) = 1{,}000{,}000{,}000 = 10^9$

1 OR 2 OR 3 say:

$P(t) = P_0 e^{kt}$, where

P_0 = initial pop.

k = relative growth rate

t = time

$P(3) = P_0 e^{3k} = 102{,}900$

$P(4) = P_0 e^{4k} = 720{,}300$

$P_0 e^{4k} - P_0 e^{3k} = 618{,}600$

Matt's idea: shift it:

$\hat{P}_0 = 102{,}900$

So

$\hat{P}(0) = 102{,}900$

$\hat{P}(1) = 720{,}300$

$\hat{P}(t) = 102{,}900 e^{kt}$

$\hat{P}(1) = 102{,}900 e^{k} = 720{,}30$

$e^{k} = \frac{720{,}300}{102{,}900} = \frac{7203}{1029}$

$k = \ln\left(\frac{7203}{1029}\right) = \ln(7)$

$\therefore\ \hat{P}(t) = 102{,}900 e^{\ln(7) t}$

$= 102{,}900 \cdot 7^{t}$

What's $P(t)$, then?

How to find the Actual $P(0)$?

$\hat{P}(-3) = 102900 \cdot 7^{-3} = 300$!

You strikes!

$$P(3) = P_0 e^{3K} = 102{,}900 \Longrightarrow P_0 = \frac{102900}{e^{3K}}$$

$$P(4) = P_0 e^{4K} = 720{,}300 \Longrightarrow P_0 = \frac{720{,}300}{e^{4K}}$$

$$P_0 = P_0$$

$$\frac{102900}{e^{3K}} = \frac{720300}{e^{4K}}$$

$$102900 e^{4K} = 720300 e^{3K}$$

$$102900 e^{4K} - 720{,}300 e^{3K} = 0$$

$$e^{3K}\left[102{,}900 e^{K} - 720{,}300\right] = 0 \Longrightarrow$$

$$102900 e^{K} - 720300 = 0$$

$$102900 e^{K} = 720300$$

$$e^{K} = \frac{7203}{1029} = 7$$

$$K = \ln(7)$$

$$\therefore P(t) = P_0 e^{\ln(7)t} = P_0 \cdot 7^t$$

To find P_0, let $t = 3$:

$$P(3) = P_0 \cdot 7^3 = 102900$$

$$P_0 = \frac{102900}{7^3} = \frac{102900}{343} = 300!$$

This is how I done it. Matt's attack was legit.

Radioactive Decay is exponential decay.

$\boxed{9.}$ The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.

(a) Find the mass that remains after t years.

(b) How much of the sample remains after 100 years?

(c) After how long will only 1 mg remain?

Aug 28-11:47 AM

10. A sample of tritium-3 decayed to 94.5% of its original amount after a year.

(a) What is the half-life of tritium-3?

(b) How long would it take the sample to decay to 20% of its original amount?

NEWTON'S LAW OF COOLING

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming.) If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant.

$$y(t) = T(t) - T_s \qquad y'(t) = T'(t) \qquad \frac{dy}{dt} = ky$$

15. When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room its temperature has increased to 10°C.

(a) What is the temperature of the drink after 50 minutes?

(b) When will its temperature be 15°C?

CONTINUOUSLY COMPOUNDED INTEREST

$$A_0(1 + r)^t \qquad\qquad A_0\left(1 + \frac{r}{n}\right)^{nt}$$

$$A(t) = \lim_{n \to \infty} A_0\left(1 + \frac{r}{n}\right)^{nt}$$

19. (a) If \$3000 is invested at 5% interest, find the value of the investment at the end of 5 years if the interest is compounded (i) annually, (ii) semiannually, (iii) monthly, (iv) weekly, (v) daily, and (vi) continuously.

(b) If $A(t)$ is the amount of the investment at time t for the case of continuous compounding, write a differential equation and an initial condition satisfied by $A(t)$.