7.4 DERIVATIVES OF LOGARITHMIC FUNCTIONS

7.4 #s 3, 6, 18, 20, 26, 27, 33, 37, 42, 47, 63, 70, 77, 81

Note $x>0$, here.

1 $$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$y = \ln x$
$e^y = x$
$e^y \cdot y' = 1$
$y' = \frac{1}{e^y}$
$y' = \frac{1}{x}$

PROOF Let $y = \ln x$. Then

Differentiating this equation implicitly with respect to x, we get

$\frac{d}{dx}[y^7] = 7y\frac{dy}{dx} = 7yy'$ by chain Rule.

$$e^y \frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$x^2y + 3xy^3 = 7$
$2xy + x^2y' + 3y^3 + 9xy^2y' = 0$
$y' = \frac{-2x + 3y^3}{x^2 + 9xy^2}$

Chain Rule in action

2 $$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

$\frac{d}{dx}[f(g(x))] = \frac{df}{dg} \cdot \frac{dg}{dx}$

Most common.

$f(x) = \ln(x)$
$g(x) = u$

2–26 Differentiate the function.

2. $f(x) = \ln(x^2 + 10)$

$$f'(x) = \frac{2x}{x^2+10}$$

$$\int \frac{2x}{x^2+10}\,dx = \ln(x^2+10) + C$$

Let $u = x^2+10$.

Then $du = 2x\,dx$

This gives $\int \frac{du}{u} = \ln|u| + C$

why absolute value?

31–34 Differentiate f and find the domain of f.

33. $f(x) = \ln(x^2 - 2x)$

$$\mathcal{D}(f) = \{x \mid x^2 - 2x > 0\} = \boxed{(-\infty, 0) \cup (2, \infty)}$$

$$x(x-2) > 0$$

$$\boxed{f'(x) = \frac{2x-2}{x^2-2x}}$$

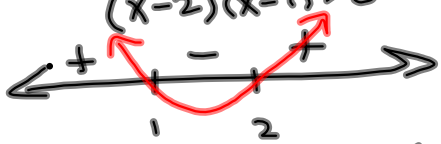
61–64 Discuss the curve under the guidelines of Section 4.5.

64. $y = \ln(x^2 - 3x + 2) = f(x)$

Recall 4.5 Guidelines:
A. Domain
B. *y*-intercept
C. Symmetry
D. Asymptotes
E. Increasing/Decreasing
F. Extremes
G. Concavity.
H. Sketch

(A) $x^2 - 3x + 2 > 0$

$(x-2)(x-1) > 0$



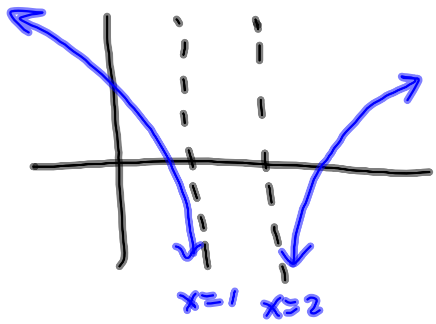
$\mathcal{D} = (-\infty, 1) \cup (2, \infty)$

(B) $f(0) = \ln(2) \rightsquigarrow (0, \ln(2))$

$f(-x) = ?$

(C) None

(D) H.A. - None. V.A.: $x = 1$, $x = 2$



(E) $f'(x) = \dfrac{2x-3}{x^2-3x+2} = \dfrac{2(x-\frac{3}{2})}{(x-2)(x-1)}$

$\underbrace{\text{Critical values:}}_{x \in \mathcal{D}(f) \text{ \& }}$ None!

either $f'(x) = 0$

or $f'(x) \nexists$

$x = \frac{3}{2} \notin \mathcal{D}(f)$

check sign of $f'(x)$ in the intervals $(-\infty, 1)$ and $(1, \infty)$ to see if inc./dec.

(E) $f'(x) = \dfrac{2x-3}{x^2-3x+2}$

$\Rightarrow$ (F) $f''(x) = \dfrac{2(x^2-3x+2)-(2x-3)(2x-3)}{(x^2-3x+2)^2}$

$= \dfrac{2x^2-6x+4-(4x^2-12x+9)}{(\quad)^2} = \dfrac{2x^2-6x+4-4x^2+12x-9}{(\quad)^2}$

$= \dfrac{-2x^2+6x-5}{(\quad)^2} = -\dfrac{2x^2-6x+5}{(\quad)^2} = -\dfrac{\overbrace{2x^2-6x+5}^{>0}}{(x^2-3x+2)^2}$

$b^2-4ac = (-6)^2-4(2)(5) = 36-40 = -4 \Rightarrow$ no real roots.

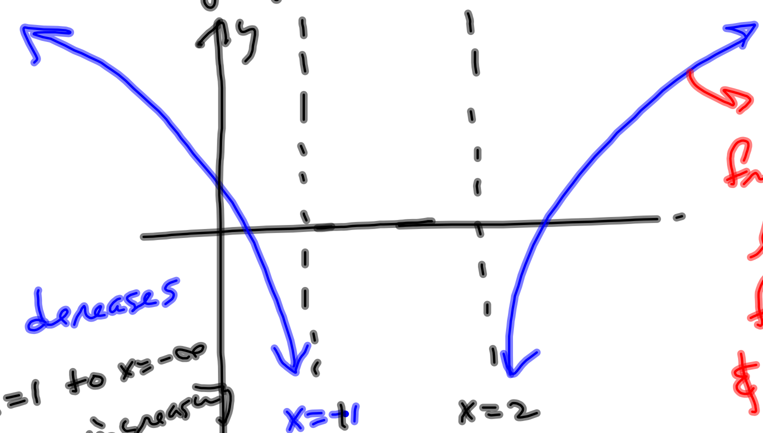
$= -\dfrac{\overbrace{2x^2-6x+5}^{>0}}{\underbrace{(x^2-3x+2)^2}_{\geq 0}}$ = negative on its domain $\Rightarrow$ Concave down.

Increasing on $(2, \infty)$

Decreasing on $(-\infty, 1)$

Picture on previous page is AOK.

At the point we found
$\mathcal{D} = (-\infty, +1) \cup (2, \infty)$ & had our
asymptotes



I got this from:
$\ln(x)$ is increasing
function, $\mathcal{D} = (0, \infty)$
& $x^2 - 3x + 2$ is increasing
from 0 to ∞
as we move
right.

As x decreases
from $x=1$ to $x=-\infty$
x^2-3x+2 is increasing
from 0 to infinity
so $\ln(x^2-3x+2)$ is
going from $-\infty$ to $+\infty$

Book Answer:

64. $y = f(x) = \ln(x^2 - 3x + 2) = \ln[(x-1)(x-2)]$ **A.** $D = \{x \text{ in } \mathbb{R} \mid x^2 - 3x + 2 > 0\} = (-\infty, 1) \cup (2, \infty)$.

B. y-intercept: $f(0) = \ln 2$; x-intercepts: $f(x) = 0 \Leftrightarrow x^2 - 3x + 2 = e^0 \Leftrightarrow x^2 - 3x + 1 = 0 \Leftrightarrow$

$x = \frac{3 \pm \sqrt{5}}{2} \Rightarrow x \approx 0.38, 2.62$ **C.** No symmetry **D.** $\lim_{x \to 1^-} f(x) = \lim_{x \to 2^+} f(x) = -\infty$, so $x = 1$ and $x = 2$ are VAs.

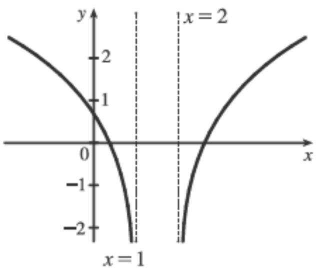
No HA **E.** $f'(x) = \dfrac{2x-3}{x^2-3x+2} = \dfrac{2(x-3/2)}{(x-1)(x-2)}$, so $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 2$. Thus, f is decreasing on $(-\infty, 1)$ and increasing on $(2, \infty)$. **F.** No extreme values

G. $$f''(x) = \frac{(x^2-3x+2)\cdot 2 - (2x-3)^2}{(x^2-3x+2)^2}$$
$$= \frac{2x^2 - 6x + 4 - 4x^2 + 12x - 9}{(x^2-3x+2)^2} = \frac{-2x^2+6x-5}{(x^2-3x+2)^2}$$

The numerator is negative for all x and the denominator is positive, so $f''(x) < 0$ for all x in the domain of f. Thus, f is CD on $(-\infty, 1)$ and $(2, \infty)$.

No IP

H.



MILLS

3

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \forall x \neq 0}$$

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x} \quad \forall x>0 \qquad \ln(|x|) = \begin{cases} \ln(x) & \text{if } x>0 \\ \ln(-x) & \text{if } x<0 \end{cases}$$

$$= \begin{cases} \ln(x) & \text{if } x>0 \\ \frac{1}{\ln(x)} & \text{if } x<0 \end{cases} = \begin{cases} \ln(x) & \text{if } x>0 \\ (\ln(x))^{-1} & \text{if } x<0 \end{cases}$$

$$\frac{d}{dx}\left[\ln|x|\right] = \begin{cases} \frac{1}{x} & \text{if } x>0 \\ \boxed{} & \end{cases}$$

Come back to this mañana

Teacher's brain is flatulent.

4

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

The difference of the logs is the log of quotient.

69–80 Evaluate the integral.

69. $\displaystyle\int_2^4 \frac{3}{x}\,dx = 3\int_2^4 \frac{dx}{x} = 3\ln|x|\Big]_2^4 = 3(\ln(4)-\ln(2))$

$= 3\ln(2)$

$u = \ln x$

$du = \frac{1}{x}dx$

74. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

$= \displaystyle\int \sin(\ln x)\cdot\frac{1}{x}\,dx = \int \sin(u)\,du = -\cos(u) + C$

$= -\cos(\ln(x)) + C$

5 $$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \frac{\sin x}{\cos x}\,dx \qquad u = \cos x \qquad du = -\sin x\,dx$$

$$= -\int \frac{1}{\cos x}\cdot(-\sin x\,dx)$$

$$= -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

$$= \ln|(\cos x)^{-1}| + C$$

$$= \ln|\sec x| + C$$

$$dx = \frac{du}{-\sin x}$$

$$\int \frac{\sin x}{u}\cdot\frac{du}{-\sin x}$$

$$= -\int \frac{du}{u}$$

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}$$

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

I always forget whether to multiply or divide by $\ln(a)$

Two things I never forget:

1. $\log_b x = \frac{\ln x}{\ln b}$ change of base

2. b^x & $\log_b x$ are inverse functions.

$$\frac{d}{dx}\left[1.2^x\right] = \ln(1.2)\cdot 1.2^x$$

$$\frac{d}{dx}\left[3^x\right] = \frac{d}{dx}\left[\left(e^{\ln(3)}\right)^x\right]$$

$$(a^b)^c = a^{bc}$$

$$= \frac{d}{dx}\left[e^{\ln(3)\cdot x}\right] = \ln(3)e^{\ln(3)\cdot x}$$

$$\frac{d}{dx}\left[e^{bx}\right] = be^{bx}$$

$$= \ln(3)\left(e^{\ln(3)}\right)^x = \ln(3)\cdot 3^x$$

$$\frac{d}{dx}\left[\log_5 x\right] = \frac{d}{dx}\left[\frac{\ln(x)}{\ln(5)}\right] =$$

$$\frac{1}{\ln(5)}\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{\ln(5)}\cdot\frac{1}{x}$$

$$\frac{d}{dx}\left[\log_7 x\right] = \frac{1}{\ln 7}\cdot\frac{1}{x}$$

$\frac{\ln(x)}{\ln(7)}$

$$\frac{d}{dx}\left[7^x\right] = (\ln 7)\cdot 7^x$$

$e^{\ln(7)x}$

$$\boxed{\frac{d}{dx}\left[e^x\right] = e^x}$$

Keeping the $\ln(7)$ upstairs or downstairs is something I re-learn every year.

MILLS

3

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \forall x \neq 0}$$

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x} \quad \forall x > 0 \qquad \ln(|x|) = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} \ln(x) & \text{if } x > 0 \\ \cancel{\frac{\ln(x)}{\ln(x)}} & \text{if } x < 0 \end{cases} = \begin{cases} \ln(x) & \text{if } x > 0 \\ \cancel{(\ln(x))^{-1}} & \text{if } x < 0 \end{cases}$$

here's the Boo-Boo.

$$\frac{d}{dx}\left[\ln|x|\right] = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \boxed{} & \end{cases}$$

Come back to this mañana

Teacher's brain is flatulent.

Brain Fart Fix:

$$\frac{d}{dx}\left[\ln(-x)\right] = \frac{-1}{-x} = \frac{1}{x}.$$ Forgeddabout the rest.

$$\frac{d}{dx}\left[\ln(-x)\right] = \frac{d}{dx}\Big[$$

Sep 1-12:10 PM

7.4 #20 $\ln\sqrt{\frac{a^2-z^2}{a^2+z^2}} = \ln\left(\left(\frac{a^2-z^2}{a^2+z^2}\right)^{\frac{1}{2}}\right)$

$= \frac{1}{2}\ln\left(\frac{a^2-z^2}{a^2+z^2}\right) = \frac{1}{2}\left[\ln(a^2-z^2) - \ln(a^2+z^2)\right]$

$\Rightarrow$ the derivative w.r.t. z is

$\frac{1}{2}\left[\frac{-2z}{a^2-z^2} - \frac{2z}{a^2+z^2}\right]$ Treat a as a constant.

The integration formula that follows from Formula 7 is

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad a \neq 1$$

$$\int 3^x\,dx = \int e^{\ln(3)x}\,dx = \frac{1}{\ln(3)}\int e^{\ln(3)x}\cdot \ln(3)\,dx$$

$u = \ln(3)x$

$du = \ln(3)\,dx$

$$= \frac{1}{\ln(3)}\int e^u \cdot du = \frac{1}{\ln(3)}e^u + C$$

$$= \frac{1}{\ln(3)}\cdot e^{\ln(3)x} + C = \frac{1}{\ln(3)}\cdot 3^x + C$$

STEPS IN LOGARITHMIC DIFFERENTIATION

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the properties of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

7.4 Due Friday.

Example 15 is a good one. I never much think of using logarithmic differentiation to ease the calculation of a messy quotient rule or product rule problem.

41–52 Use logarithmic differentiation to find the derivative of the function.

$$\frac{d}{dx}\left[\ln(u(x))\right] = \frac{u'(x)}{u(x)}$$

41. $y = (2x+1)^5(x^4-3)^6$

$$\ln(y) = \ln\left((2x+1)^5(x^4-3)^6\right)$$

$$= 5\ln(2x+1) + 6\ln(x^4-3)$$

$$\frac{y'}{y} = 5\cdot\frac{2}{2x+1} + 6\cdot\frac{4x^3}{x^4-3}$$

$$y' = \left[\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right](2x+1)^5(x^4-3)^6$$

Old way: $y' = 5(2x+1)^4(2)(x^4-3)^6 + (2x+1)^5(6)(x^4-3)^5(4x^3)$

To *me*, the big application is differentiating functions that involve variable functions to variable powers.

Bird finds a new perch

50 $\boxed{y = (\sin x)^{\ln x}}$

$$\ln(y) = \ln\left((\sin(x))^{\ln(x)}\right) = \ln(x)\ln(\sin(x)) \Longrightarrow$$

$$\frac{y'}{y} = \frac{1}{x}\ln(\sin(x)) + \ln(x)\cdot\frac{\cos(x)}{\sin(x)} \Longrightarrow$$

$$y' = \left[\frac{\ln(\sin(x))}{x} + \frac{\ln(x)\cos(x)}{\sin(x)}\right](\sin(x))^{\ln(x)}$$

Your textbook also uses logarithmic differentiation for an alternate proof of the power rule.

Another cool application is defining the number e as a limit. See discussion at the bottom of page 418 thru top of page 419.

8 $$e = \lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

same deal.

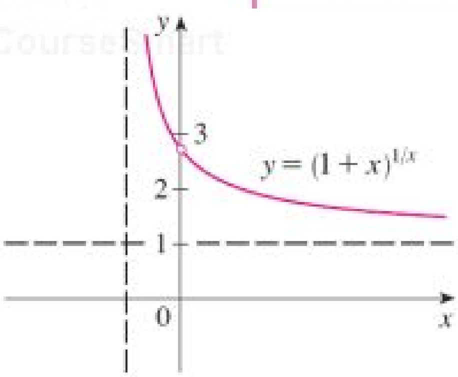


FIGURE 6

Aug 27-11:33 AM

#77

$$\cos^2 x = 1 - \sin^2 x = \frac{1 + \cos(2x)}{2}$$