95. (a) Use mathematical induction to prove that for $x \geqslant 0$ and any positive integer n,

$$e^x \geqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

(b) Use part (a) to show that $e > 2.7$.

(c) Use part (a) to show that

$$\lim_{x \to \infty} \frac{e^x}{x^k} = \infty$$

for any positive integer k.

Recall:
Last time we started this proof, but nobody remembered induction process.

Part (a)

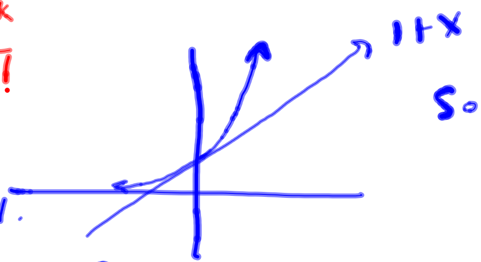
We prove this on the next page(s).

Claim: $\forall x \geq 0$,

$$e^x \geq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \quad \forall n \in \mathbb{N}$$

Proof $\quad \sum_{k=0}^{n} \frac{x^k}{k!}$

$e^x \geq 1+x$



So $e^x \geq 1+x \ \forall x$

So, true for $n=1$.

Suppose it's true for some $k \geq 1$.
We show it holds for $k+1$.

$$\frac{d}{dx}\left[\frac{x^{k+1}}{(k+1)!}\right] = (k+1)\frac{x^k}{(k+1)!} = \frac{\cancel{(k+1)}\ x^k}{\cancel{(k+1)}\underbrace{(k)(k-1)\cdots(3)(2)(1)}_{k!}}$$

$$f(x) = e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \cdots - \frac{x^{k+1}}{(k+1)!}$$

$$f'(x) = e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^k}{k!} \geq 0,$$

since $e^x \geq 1 + x + \cdots + \frac{x^k}{k!}$, by assumption.

PAUSE.

So $f(x) = e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{k+1}}{(k+1)!}$ is non-decreasing.

What happens @ $x=0$?

$f(0) = e^0 - 1 - 0 - 0 - 0 \cdots = 1 - 1 = 0$

$f(0)=0$, $f(x)$ is increasing.

$\therefore$ $0 \leq f(x)$ $\forall x$.

This means

$f(x) = e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{k+1}}{(k+1)!} \geq 0$

$\Rightarrow e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k+1}}{(k+1)!}$

So our claim holds for $n = k+1$, whenever it holds for $n = k$.

$\therefore$ the claim holds $\forall n \in \mathbb{N}$ ∎

Our Goal: To show that exponential growth exceeds polynomial growth.

x^5 x^n , 3^x

Aug 26-12:19 PM

To convince you, we show that

$$\frac{e^x}{x^n} \xrightarrow{x \to \infty} \infty, \quad \forall n \in \mathbb{N}$$

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty.$$

Exponential Functions grow faster (eventually) than any power function.

Let k be fixed.

Then $(k+1)!$ is just a fixed number.

$k = 4 \Rightarrow (k+1)! = 5! = 120$

$$\lim_{x \to \infty} \frac{x^4}{5!} = = \frac{1}{120} \lim_{x \to \infty} x^4 = \infty.$$

what's $\lim_{x \to \infty} \frac{x^n}{(n+1)!}$

Claim: $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$, $\forall n \in \mathbb{N}$.

$3+2 \geq 2$

Proof:

By part (a), $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}$

for all $x \geq 0$, and for all $k \in \mathbb{N}$.

$$\frac{e^x}{x^k} \geq \overbrace{\frac{1}{x^k} + \frac{1}{x^{k-1}} + \cdots + \frac{x^k}{k!\,x^k}}^{3} \overset{+}{} \overbrace{\frac{x^{k+1}}{(k+1)!\,x^k}}^{2}$$

$k+1-k=1$

$$\geq \frac{x^{k+1}}{(k+1)!\,x^k} = \frac{x}{(k+1)!} \xrightarrow{x\to\infty} \infty.$$

i.e. $\boxed{\frac{e^x}{x^k} \xrightarrow{x\to\infty} \infty.}$

Use (2) to show that $e > 2.7$

'e' for 'Euler', who discovered it.

Show $e > 2.7$

$$e^1 \geq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$$

$$\underline{e^1} \geq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = \frac{65}{24} = 2.708\overline{3} > \underline{2.7}$$

$$2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{65}{24}$$

Attachments

5-1-spread.xlsx

cosine-animation-riemann.wmf

5-1-spread-for-lecture.xlsx