§7.1 # 42

$G(x) = \frac{1}{f^{-1}(x)}$ $\qquad f(3) = 2 \qquad \left(f^{-1}(2) = 3\right)$

$f'(3) = \frac{1}{9}$

Find $G'(2)$

$G(x) = \left(f^{-1}(x)\right)^{-1}$ → Reciprocal (Arithmetic)

→ Inverse Function (Composition)

$G(x) = \left(f^{-1}(x)\right)^{-1}$

$G'(x) = -\left(f^{-1}(x)\right)^{-2} \cdot \left(f^{-1}\right)'(x)$

$\Rightarrow G'(2) = -\left(f^{-1}(2)\right)^{-2} \cdot \left(f^{-1}\right)'(2)$

$= -\left(3^{-2}\right)\left(\frac{1}{f'(f^{-1}(2))}\right) = -\left(3^{-2}\right)\left(\frac{1}{f'(3)}\right)$

$= -\left(3^{-2}\right)\left(\frac{1}{\left(\frac{1}{9}\right)}\right) = -\frac{1}{9} \cdot \frac{1}{\left(\frac{1}{9}\right)} = -\frac{1}{9} \cdot \frac{9}{1} = -1$

## 7.2 EXPONENTIAL FUNCTIONS AND THEIR DERIVATIVES

Recall: In general, an **exponential function** is a function of the form

$$\left.\begin{aligned} \cos x &= \frac{e^{ix}+e^{ix}}{2} \\ \sin x &= \frac{e^{ix}-e^{-ix}}{2i} \end{aligned}\right\} \text{Ignore} \qquad f(x) = a^x$$

If $x = n$, a positive integer, then

$$x^3 = \underbrace{x \cdot x \cdot x}_{3\text{ of 'em}} \qquad a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

If $x = 0$, then $a^0 = 1$, and if $x = -n$, where $n$ is a positive integer, then

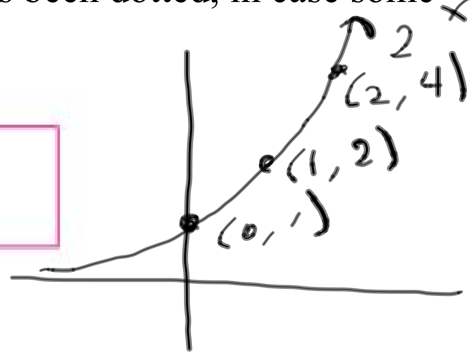
$$a^{-n} = \frac{1}{a^n}$$

If $x$ is a rational number, $x = p/q$, where $p$ and $q$ are integers and $q > 0$, then

$$3^{\frac{2}{3}} = \sqrt[3]{3^2} \qquad a^x = a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

Page 393 discussion is to motivate/justify the notion of using IRRATIONAL exponents, by reasoning from integers to rationals thence to irrationals. This is mainly to convince you that it makes sense to think of exponential functions, whose domain is all real numbers.

In practice, students prob'ly don't need these justifications, but it makes mathematicians feel better knowing that this *i* has been dotted, in case some genius reads the book.




1 $$a^x = \lim_{r \to x} a^r \qquad r \text{ rational}$$

Continuity justification
No gaps.

**2 THEOREM** If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. In particular, $a^x > 0$ for all $x$. If $0 < a < 1$, $f(x) = a^x$ is a decreasing function; if $a > 1$, $f$ is an increasing function. If $a, b > 0$ and $x, y \in \mathbb{R}$, then

1. $a^{x+y} = a^x a^y$ 2. $a^{x-y} = \dfrac{a^x}{a^y}$ 3. $(a^x)^y = a^{xy}$ 4. $(ab)^x = a^x b^x$



$$3^{2+5} = 3^7 = 3^2 \cdot 3^5$$

$$\frac{3^2}{3^5} = 3^{2-5} = 3^{-3} = \frac{1}{27}$$

$$(a^x)^y = a^{xy} \qquad (2^5)^7 = 2^{35}$$

$$(2 \cdot 7)^3 = 2^3 \cdot 7^3$$

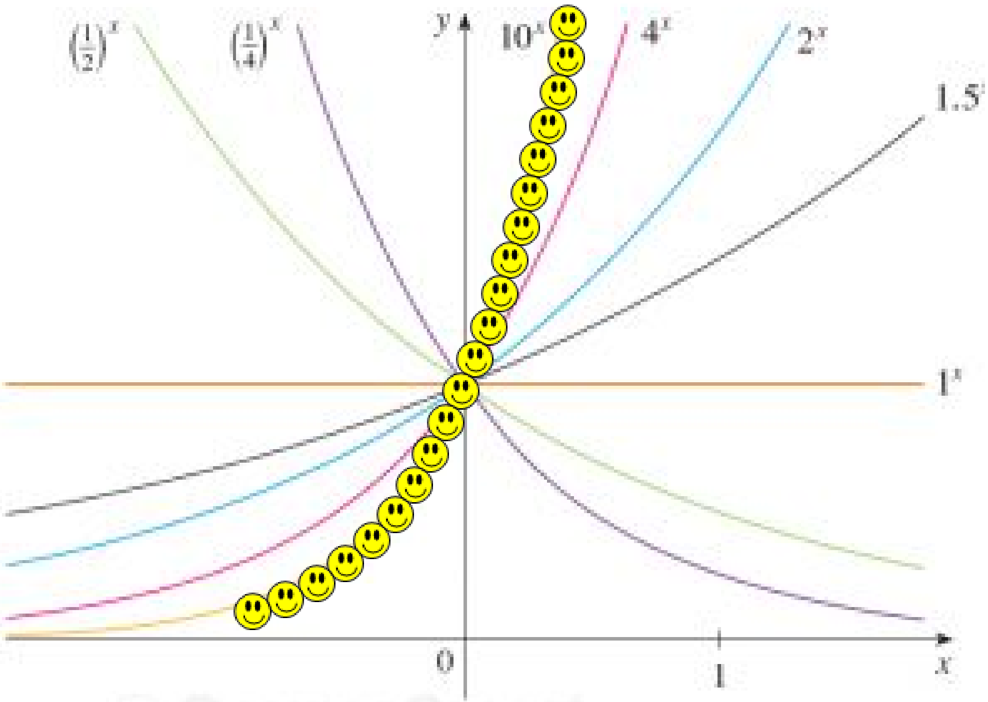





**FIGURE 3**
Members of the family of exponential functions

Figure 4 shows

Exponentials grow faster than polynomials

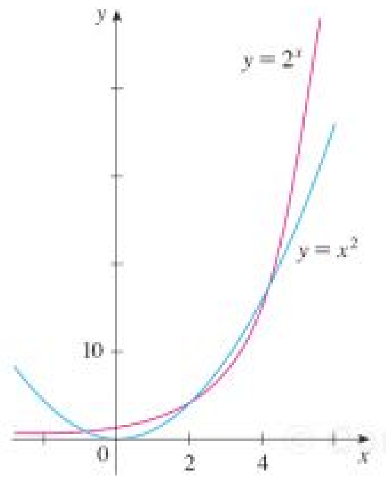



FIGURE 4

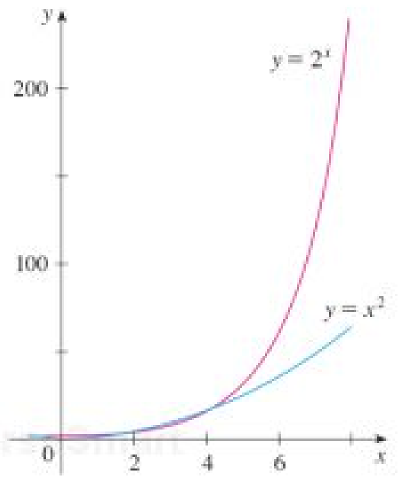



FIGURE 5

Comparing power function (and polynomial) behavior to exponential function behavior.

**3** If $a > 1$, then $\lim_{x \to \infty} a^x = \infty$ and $\lim_{x \to -\infty} a^x = 0$

If $0 < a < 1$, then $\lim_{x \to \infty} a^x = 0$ and $\lim_{x \to -\infty} a^x = \infty$

e.g. $\left(\frac{1}{2}\right)^x$

$.732^x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \boxed{\lim_{h \to 0} \frac{a^x(a^h - 1)}{h}}$$

See pp 396-7

$$= \lim_{h \to 0} \left[ a^x \left( \frac{a^h - 1}{h} \right) \right] = a^x \lim_{h \to 0} \left[ \frac{a^h - 1}{h} \right]$$

You made that up!

**7 DEFINITION OF THE NUMBER $e$**

$e$ is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

Magic Number.

$e^h \xrightarrow{h \to 0} 1$ exactly as fast as $h \to 0$.

$$\lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$$

**8 DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION**

e is chosen so its height is its slope

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$


$\frac{d}{dx}\left[e^x\right]\Big|_{x=0} = 1$

$\frac{d}{dx}\left[e^{5x^2-10}\right]$

9 $= \left[e^{5x^2-10}\right] \cdot 10x$

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

**10 PROPERTIES OF THE NATURAL EXPONENTIAL FUNCTION** The exponential function $f(x) = e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus $e^x > 0$ for all $x$. Also

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$$

$2 < e < 3$

So the $x$-axis is a horizontal asymptote of $f(x) = e^x$.

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$$\boxed{\int e^x\,dx = e^x + C}$$

Differentiate:

(b) $f(x) = \dfrac{1}{1-e^x}$ $= (1-e^x)^{-1}$

$\Longrightarrow f'(x) = \boxed{-(1-e^x)^{-2}(-e^x)} = \dfrac{e^x}{(1-e^x)^2}$

**31–46** Differentiate the function.

**38.** $f(t) = \sin(e^t) + e^{\sin t}$ $= \sin(e^t) + e^{\sin(t)}$

$\Longrightarrow f'(t) = \cos(e^t)\cdot e^t + e^{\sin(t)}\cdot\cos(t)$

Aug 25-11:53 AM

**53.** For what values of $r$ does the function $y = e^{rx}$ satisfy the equation $y'' + 6y' + 8y = 0$?

Characteristic Equation:

Let D stand for $\frac{d}{dx}$

Then $\frac{dy}{dx} = Dy$

$D^2 y = \frac{d^2 y}{dx^2} = y''$

$D^2 y + 6Dy + 8y = 0$

$(D^2 + 6D + 8)y = 0 \implies D^2 + 6D + 8 = 0$ or $y = 0$

$(D+4)(D+2) = 0 \implies D = -4, -2$

Let $r = -4$

**53.** For what values of $r$ does the function $y = e^{rx}$ satisfy the equation $y'' + 6y' + 8y = 0$?

Let $r = -4$. Then $y = e^{-4x}$ will do it!

$y' = -4e^{-4x}$

$y'' = 16e^{-4x}$

Differential Equation: Find function from info on its "movements"

$y'' + 6y' + 8y =$

$16e^{-4x} + 6(-4e^{-4x}) + 8e^{-4x}$

$= 16e^{-4x} - 24e^{-4x} + 8e^{-4x} = 0.$

So $y = e^{-4x}$ satisfies the D.E.

$\boxed{75.}$ $\int e^x\sqrt{1 + e^x}\,dx$

**95.** (a) Use mathematical induction to prove that for $x \geqslant 0$ and any positive integer $n$,

$$e^x \geqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$



(b) Use part (a) to show that $e > 2.7$.

(c) Use part (a) to show that

$$\lim_{x \to \infty} \frac{e^x}{x^k} = \infty$$

for any positive integer $k$.

Attachments

5-1-spread.xlsx

cosine-animation-riemann.wmf

5-1-spread-for-lecture.xlsx