Final Test Survey

Test 1:

(1) Let $f(x) = x^2 - 3x$. Find $\frac{df}{dx}$ in two ways:

(a) Limit definition of the derivative

(b) The easy way

(a) $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 - 3(x+h) - (x^2-3x)}{h} = \frac{\cancel{x^2} + 2xh + h^2 \cancel{-3x} - 3h \cancel{-x^2} \cancel{+3x}}{h}$

$= \frac{2xh + h^2 - 3h}{h} = \frac{h(2x+h-3)}{h} = 2x+h-3 \xrightarrow{h \to 0} 2x-3$

(b) $\frac{d}{dx}(x^2-3x) = 2x-3$

(2) Find an equation of the tangent line to $f(x) = x^2 - 3x$ at the point $(2,-2)$.

$$f'(x) = 2x - 3$$

$$f'(2) = 2(2) - 3 = 1 = m$$

$$y = m(x - x_1) + y_1$$

$$\boxed{y = 1(x-2) - 2}$$

(3) Evaluate the following limits

(a) $\lim_{x\to 2} \frac{x^2+5x-14}{2x^2-9x+10}$.

$$\frac{x^2+5x-14}{2x^2-9x+10} = \frac{(x+7)\cancel{(x-2)}}{\cancel{(x-2)}(2x-5)} = \frac{x+7}{2x-5} \xrightarrow{x\to 2} \frac{2+7}{2(2)-5} = \frac{9}{-1} = -9$$

$(x \neq 2)$

(b) $$\lim_{x\to 3^-} \frac{|x-3|}{2x^2+x-15} = \lim_{x\to 3^-} \frac{-(x-3)}{2x^2+x-15} = \lim_{x\to 3^-} \frac{-\cancel{(x-3)}^{1}}{\cancel{(x-3)}_{1}(2x+5)} = \lim_{x\to 3^-} \frac{-1}{2x+5} = \boxed{\frac{-1}{11}}$$

$$|x-3| = \begin{cases} x-3 & \text{if } x-3 \geq 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x \geq 3 \\ -(x-3) & \text{if } x < 3 \end{cases}$$

$$2x^2-6x+5x-15 = 2x(x-3)+5(x-3) = (x-3)(2x+5)$$

(c) $\lim_{x\to 3} \frac{|x-3|}{2x^2+x-15}$ $= f(x)$ $\nexists$ $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$

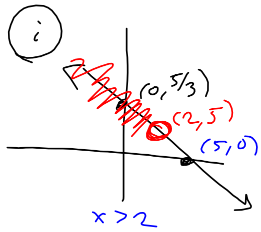
$$\lim_{x\to 3^+} \frac{|x-3|}{2x^2+x-15} = \lim_{x\to 3^+} \frac{x-3}{(x-3)(2x+5)} = \frac{1}{11} \neq -\frac{1}{11}$$

(4) Let $f(x) = \begin{cases} -\frac{5}{3}x + \frac{25}{3} & x > 2 \quad \text{(i)} \\ x^2 + 2x - 3 & x \le 2 \quad \text{(ii)} \end{cases}$

(a) Sketch the graph of f. Show all intercepts and local extremes, if any

(b) On what interval(s) is f cnts?

(i)



$-\frac{5}{3}x + \frac{25}{3} = 0$

$-5x + 25 = 0$

$-5x = -25$

$x = 5$

$-\frac{5}{3}(2) + \frac{25}{3} = \frac{-10+25}{3} = \frac{15}{3} = 5$

(ii) $x^2 + 2x - 3 = y$ (1,0), (-3,0)

$= (x+3)(x-1)$

$y' = 2x + 2 \overset{SET}{=} 0$

$2x = -2$

$x = -1$

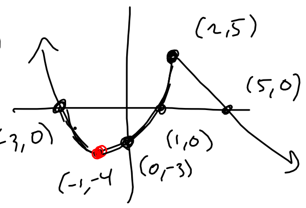
$(-1)^2 + 2(-1) - 3$

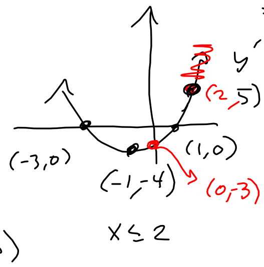
$= 1 - 2 - 3 = -4$

$2^2 + 2(2) - 3$

$= 4 + 4 - 3 = 5$

(a)





(b) f is continuous on $(-\infty, \infty)$

(5) Prove that $\lim_{x\to 3}(2x-5)=1$, by the formal def'n of limit

Scratch

want $|2x-5-1|<\varepsilon$

$|2x-6|<\varepsilon$

$2\underbrace{|x-3|}_{<\delta}<\varepsilon$

$|x-3|<\frac{\varepsilon}{2}=\delta$

Proof Let $\varepsilon>0$ be given.

Define $\delta=\frac{\varepsilon}{2}$. Then $0<|x-3|<\delta \Rightarrow$

$|2x-5-1|=|2x-6|=2|x-3|<2\delta=2\cdot\frac{\varepsilon}{2}$

$=\varepsilon$ ■

Bonus Prove that $\lim_{x\to 2}(x^2+2x-3)=5$, by the formal def'n of limit

Want $|x^2+2x-3-5|<\varepsilon$

$|x^2+2x-8|$

$=\underbrace{|x+4|}\underbrace{|x-2|}_{<\delta}$

Need a bound on $|x+4|$

Assume $\delta\le 1$

Then $|x-2|<\delta\le 1$

$|x-2|<1$

$-1<x-2<1$

$+6 \quad = \quad +6 \quad = 6$

$5<x+4<7$

$\Rightarrow |x+4|<7 \rightsquigarrow \frac{\varepsilon}{7}$

Proof Let $\varepsilon>0$ be given

Define $\delta=\min\{1,\frac{\varepsilon}{7}\}$. Then

$0<|x-2|<\delta \Rightarrow$

$|x^2+2x-3-5|=|x^2+2x-8|$

$=|x+4||x-2|<7\cdot\delta\le 7\cdot\frac{\varepsilon}{7}$

$=\varepsilon$ ■

(6) Convince me that $2x^3 - 5x^2 - 19x + 42 = f(x)$
has a zero in the interval $(0,3)$,
without actually finding it

$f(0) = 42$ So, $f(3) < 0 < f(0)$, so $\exists\ c \in (0,3) \ni$
$f(3) = -6$ $f(c) = 0$ by IVT (and the fact that polynomials are continuous everywhere).

$$\begin{array}{c|cccc} 3 & 2 & -5 & -19 & 42 \\ & & 6 & 3 & -48 \\ \hline & 2 & 1 & -16 & -6 \end{array}$$

Bonus Find all zeros of $2x^3-5x^2-19x+42$ and split it into linear factors. (Stock college algebra question)

$2 \overline{)42}$
$3 \overline{)21}$
7

We know there's one between $x=0$ & $x=3$

I'm guessing $x=2$.

$\frac{p}{q}: \frac{42}{2}$

$\pm 1, \pm \frac{1}{2}$
$\pm 2, \pm \frac{2}{2}$
$\pm 3, \pm \frac{3}{2}$

$$\begin{array}{r|rrrr} 2 & 2 & -5 & -19 & 42 \\ & & 4 & -2 & -42 \\ \hline & 2 & -1 & -21 & 0 \end{array}$$

$(x-2)(2x^2-x-21)$

$a=2, b=-1, c=-21$

$b^2-4ac = (-1)^2-4(2)(-21)$
$= 1+168 = 169$

$x = \frac{1 \pm 13}{2(2)} = \frac{1 \pm 13}{4}$ ↗ $\frac{14}{4} = \frac{7}{2}$ ↘ $\frac{-12}{4} = -3$

$2(x-2)(x-\frac{7}{2})(x+3) = f(x)$

Test 2:

$2x^{4/5} \rightsquigarrow$

$\frac{2}{5} - 1 = -\frac{3}{5}$

7. Find $\frac{dy}{dx} = y'$

a) $x^2 + x^{-4} - 2\sqrt[5]{x^2} - \pi = y$

$\Longrightarrow y' = 2x - 4x^{-5} - \frac{4}{5}x^{-3/5}$

b) $\sec(3x) = y$

$y' = (\sec(3x)\tan(3x))(3)$

$(fg)' = f'g + fg'$

c) $3\sec(3x)\tan(3x) = y$

$y' = 3\left[(\sec(3x)\tan(3x))(3)\tan(3x) + \sec(3x)(\sec^2(3x))(3)\right]$

d) $\frac{\csc(x) - 2x}{x^2 + \tan(x)} = y$

$y' = \frac{(-\csc(x)\cot(x) - 2)(x^2 + \tan(x)) - (\csc(x) - 2x)(2x + \sec^2(x))}{(x^2 + \tan(x))^2}$

e) $\sqrt{(\csc(x) - 2x)^3} = y = (\csc(x) - 2x)^{\frac{3}{2}} \Longrightarrow$

$y' = \frac{3}{2}(\csc(x) - 2x)^{\frac{1}{2}}(-\csc(x)\cot(x) - 2)$

f) $\sin^2(x^3 - \tan x) = y = (\sin(x^3 - \tan(x)))^2$

$y' = \boxed{2\sin(x^3 - \tan(x))'}(\cos(x^3 - \tan(x)))(3x^2 - \sec^2(x))$

$\frac{d}{dx}[f(g(h(x)))] = \frac{df}{dg} \cdot \frac{dg}{dh} \cdot \frac{dh}{dx}$

g) $xy^2 - 3x^3y = \sin(xy)$

$1y^2 + x \cdot 2yy' - 9x^2y - 3x^3y' = (\cos(xy))(1y + xy') = y\cos(xy) + xy'\cos(xy)$

$2xyy' - 3x^3y' - xy'\cos(xy) = y\cos(xy) - y^2 + 9x^2y$

$y'(2xy - 3x^3 - x\cos(xy)) = y\cos(xy) - y^2 + 9x^2y$

$y' = \frac{y\cos(xy) - y^2 + 9x^2y}{2xy - 3x^3 + x\cos(xy)}$

(8) (a) Use a tangent line to approximate $\sin(50^\circ)$.

(b) Draw the picture that illustrates what you did.

(c) Why does this work? Smooth curves are locally linear

The Earth is flat!

(a) $\sin 50^\circ$ $\quad f(x) = \sin x$ Convert to radians!

$\sin(45^\circ) = \frac{1}{\sqrt{2}}$ $\quad x_1 = 45^\circ = \frac{\pi}{4}$

$\Delta x = 5^\circ = (5^\circ)\left(\frac{\pi \text{ rad}}{180^\circ}\right) = \frac{5\pi}{180} = \frac{\pi}{36} = \Delta x$

$y = m(x - x_1) + y_1$ $\quad x - x_1 = 50^\circ - 45^\circ = 5^\circ = \frac{\pi}{36}$

$= f'(x_1)(x - x_1) + f(x_1)$

$f'(x) = \cos x$

$f'(45^\circ) = \cos(45^\circ) = \frac{1}{\sqrt{2}}$ $\quad = \frac{1}{\sqrt{2}}\left(\frac{\pi}{36}\right) + \frac{1}{\sqrt{2}} = \frac{\pi}{36\sqrt{2}} + \frac{36}{36\sqrt{2}} = \frac{\pi + 36}{36\sqrt{2}}$

(9) Related Rates Question, like

(a) lighthouse question

(b) how fast a kite is moving away from you if you know its height and speed.

$\boxed{\text{150430-review in notes}}$

(c) How fast the area of an oil spill is changing if you know how fast its radius is growing

(d) How fast the radius ~~of~~ or surface area of a sphere is growing if you know the rate at which air is being pumped in

(e) How fast the water level is rising (falling) at a particular moment if you know the rate at which water's being pumped in (out), Maybe a cylinder. Maybe a cone.

10 Differentials: Estimate the error in a calculation based on the error in one of the measurements. Like...

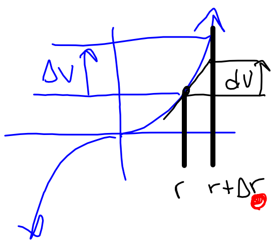
See 150430-review in notes.

a Estimate error in volume of sphere, given error in radius $\rightarrow \Delta r = dr$

b " " " Surface area of a disc, " " " "

c Estimate volume of paint to cover a surface (if sphere, a & c are same question!)

$$\Delta V = V_2 - V_1 \approx \frac{dV}{dr}\cdot \Delta r = \frac{dV}{dr}\cdot dr = dV$$



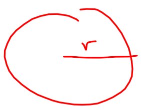
a $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$dV = 4\pi r^2 dr$

Error Estimate
OR
Change in volume, given small change in radius

b



$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$dA = 2\pi r\, dr \approx \Delta A$

$\Delta A = \pi (r+\Delta r)^2 - \pi r^2$

$\approx 2\pi r\, dr = 2\pi r \Delta r$

Test 3:

1. Sketch the graph of $f(x)=2x^3-3x^2-72x+73$

$$\begin{array}{r|rrrr} 1 & 2 & -3 & -72 & 73 \\ & & 2 & -1 & -73 \\ \hline & 2 & -1 & -73 & 0 \end{array}$$

$(x-1)(\underline{2x^2-x-73})$

$a=2,\ b=-1,\ c=-73$

$b^2-4ac=(-1)^2-4(2)(-73)$

$=1+8(73)$

$=585$

$\sqrt{585}=3\sqrt{65}$

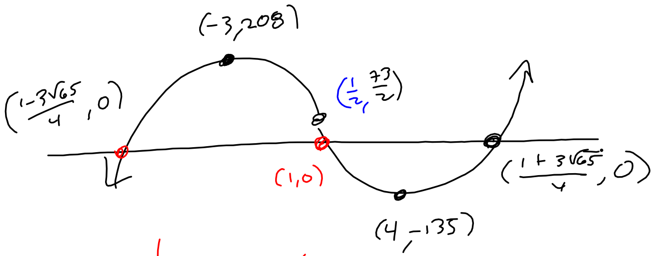
$x=\dfrac{1\pm 3\sqrt{65}}{2(2)}=\dfrac{1\pm 3\sqrt{65}}{4}$

$3 \,|\, 585$, $3 \,|\, 195$, $5 \,|\, 65$, 13

$730 - 146 = 584$ $\qquad 73 \cdot 8 = 584$ ✓

73

$1,\ \dfrac{1+3\sqrt{65}}{4},\ \dfrac{1-3\sqrt{65}}{4}$



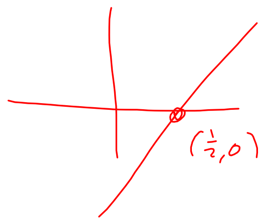
$f'(x)=6x^2-6x-72$

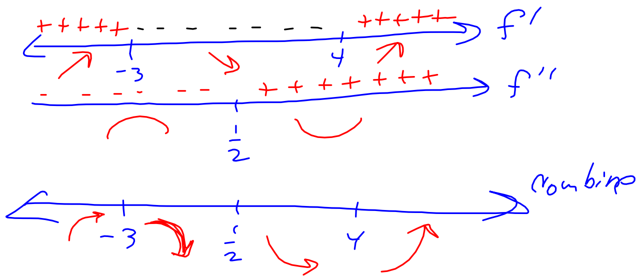
$=6(x^2-x-12)$

$=6(x-4)(x+3)\overset{SET}{=}0$

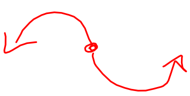
$x\in\{-3,4\}$

$f''(x)=12x-6\overset{SET}{=}0 \Rightarrow x=\frac{1}{2}$





$$\begin{array}{r|rrrr} -3 & 2 & -3 & -72 & 73 \\ & & -6 & 27 & 135 \\ \hline & 2 & -9 & -45 & 208 = f(-3) \end{array}$$



$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -3 & -72 & 73 \\ & & 1 & -1 & -\frac{73}{2} \\ \hline & 2 & -2 & -73 & \frac{73}{2} = f\left(\frac{1}{2}\right) \end{array}$$

$$\begin{array}{r|rrrr} 4 & 2 & -3 & -72 & 73 \\ & & 8 & 20 & -208 \\ \hline & 2 & 5 & -52 & -135 = f(4) \end{array}$$

(2) Check that $f(x) = x^3 - 3x^2 + 2x$ satisfies hypotheses of MVT on $[0,3]$.
Then find $c \in (0,3)$ such that $f'(c) = m_{avg}$ on $[0,3]$.
This is MVT for derivatives
There's another MVT for integrals in §5.5

Need:
cnts on $[0,3]$
difbl on $(0,3)$
f is poly $\Rightarrow$ good to go.

$$m_{AVG} = \frac{f(3)-f(0)}{3-0} = \frac{27-27+2(3)-0}{3}$$

$$= \frac{6}{3} = 2 = m_{AVG}$$

$$f'(x) = 3x^2 - 6x + 2 \overset{SET}{=} m_{avg} = 2$$

$$\Rightarrow \quad 3x^2 - 6x = 0$$

$$3x(x-2) = 0 \rightarrow c = 0, 2$$

$0 \notin (0,3)$ so $\boxed{c = 2}$

(3) Find all local extrema in $(0, 2\pi)$ for
$f(x) = -2\sin(x)\cos(x) - x$ See test 3!

(4) Compute the limits

(a) $\lim_{x\to\infty} \frac{4x^3-7x+2}{3-5x^3} = \lim_{x\to\infty} \frac{4x^3}{-5x^3} = -\frac{4}{5}$

(b) $\lim_{x\to\infty} \left(\sqrt{16x^2-5x} - 4x\right)$

(c) $\lim_{x\to\infty} \frac{x^3-5x^2+7}{3x-5x^4} = 0$

Proper Rational func.

$\frac{\text{smaller degree}}{\text{bigger degree}} \xrightarrow{x\to\infty} 0$

(5) Find the oblique asymptote for $R(x) = \dfrac{2x^3-5x+11}{x^2+7x+2}$

$$\begin{array}{r} 2x-14 \\ x^2+7x+2 \,\big)\, \overline{2x^3+0x^2-5x+11} \\ -(2x^3+14x^2+4x) \\ \hline -14x^2-9x \end{array}$$

$\boxed{y = 2x-14}$

(6) Sketch $f(x) = \dfrac{2x+3}{x-1}$. Show all intercepts, asymptotes & use 1st and 2nd derivative to analyze increasing/decreasing intervals and concavity.

Test 3 & Bonus on Test 4

(7) Find x-value where vertical distance between $f(x) = 2x^2+12x+21$ & $g(x) = x^2+2x-6$ is minimized

Minimize $|2x^2+12x+21-(x^2+2x-6)|$

$= |x^2+10x+27|$

minimize it! If it's never zero, then its vertex will do it

$x = -\dfrac{b}{2a} = -\dfrac{10}{2} = -5$

$a=1, b=10, c=27$

$b^2-4ac = 10^2-4(1)(27)$

$= 100-108 = -8$

Never touch!

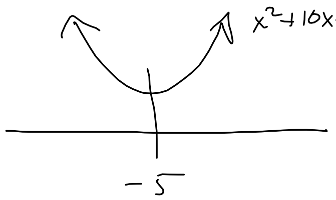
$x^2+10x+27$ has no real zeros.

$\therefore$ it lives entirely Above x-axis

$f'(x) = 2x+10 \overset{SET}{=} 0$

$2x = -10$

$\boxed{x=-5}$



$x^2+10x+27 = |x^2+10x+27|$

Bonus What is the exact minimum distance in #7?

Test 4: See Test 4 materials

Test 5: see Test 5 "

$f(-5)$

$$\begin{array}{r|rrr} -5 & 1 & 10 & 27 \\ & & -5 & -25 \\ \hline & 1 & 5 & 2 \end{array}$$

$\boxed{2 = \text{Distance}}$