§6.3

1 $$\log_a x = y \iff a^y = x$$

$\log_3 81 = 4$ because $3^4 = 81$

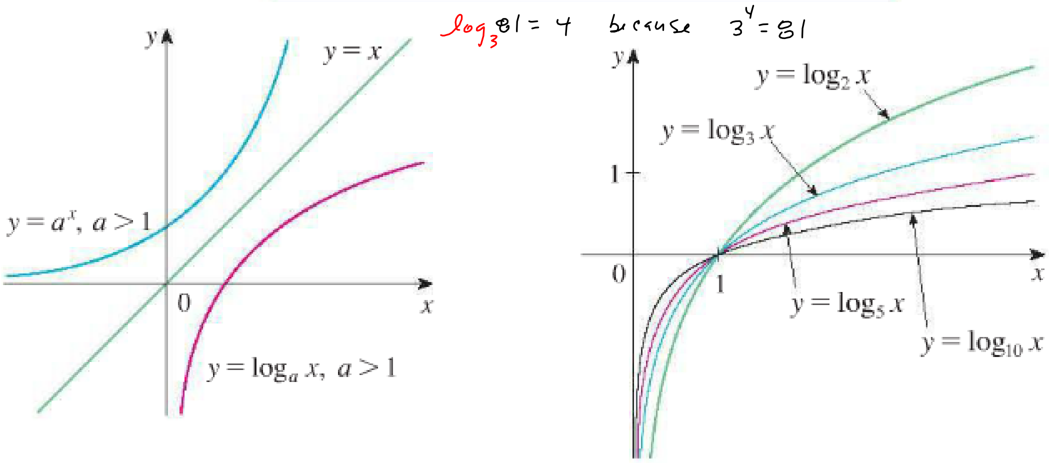


FIGURE 1

2 $$\log_a(a^x) = x \quad \text{for every } x \in \mathbb{R}$$
$$a^{\log_a x} = x \quad \text{for every } x > 0$$

$\frac{d}{dx}\left[e^{7x}\right] = 7e^{7x}$

$3^x = e^{\ln(3^x)} = 3^x$

$= e^{x\ln(3)} = e^{(\ln(3))x}$

$\frac{d}{dx}\left[3^x\right] = (\ln(3))e^{(\ln 3)x}$

$= (\ln 3)e^{x\ln 3}$

$= (\ln 3)e^{\ln(3^x)}$

$= \ln(3)3^x$

$= (\ln(3)) \cdot 3^x$

2

$$\log_a(a^x) = x \quad \text{for every } x \in \mathbb{R}$$

$$a^{\log_a x} = x \quad \text{for every } x > 0$$

3 Theorem If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y > 0$ and r is any real number, then

$\sqrt{x} = x^{\frac{1}{2}}$

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^r) = r \log_a x$

$$\boxed{\ln(5\sqrt{x})} = \ln(5) + \ln(\sqrt{x})$$
$$= \ln(5) + \ln(x^{\frac{1}{2}})$$
$$= \boxed{\ln(5) + \tfrac{1}{2}\ln(x)}$$

$$\ln\left(\frac{x-2}{x+3}\right) = \ln(x-2) - \ln(x+3)$$

9–12 Use the properties of logarithms to expand the quantity.

9. $\ln \sqrt{ab}$

10. $\log_{10} \sqrt{\dfrac{x-1}{x+1}}$ $= \log\left(\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}\right)$

$= \frac{1}{2} \log\left(\frac{x-1}{x+1}\right)$

$= \frac{1}{2}\left[\log(x-1) - \log(x+1)\right]$

$= \frac{1}{2}\log(x-1) - \frac{1}{2}\log(x+1)$

11. $\ln \dfrac{x^2}{y^3 z^4}$

12. $\ln\left(s^4 \sqrt{t\sqrt{u}}\right)$

13–18 Express the quantity as a single logarithm.

13. $2 \ln x + 3 \ln y - \ln z$

14. $\log_{10} 4 + \log_{10} a - \frac{1}{3}\log_{10}(a+1)$

15. $\ln 5 + 5 \ln 3$

16. $\ln 3 + \frac{1}{3} \ln 8$

17. $\frac{1}{3}\ln(x+2)^3 + \frac{1}{2}\left[\ln x - \ln(x^2+3x+2)^2\right]$

18. $\ln(a+b) + \ln(a-b) - 2 \ln c$

$\ln(x+2)^3 \overset{?}{=}$ $\ln((x+2)^3)$ or $(\ln(x+2))^3$

17 $\frac{1}{3}\ln\left((x+2)^3\right) + \frac{1}{2}\left[\ln(x) - \ln\left((x^2+3x+2)^2\right)\right]$

$= \ln\left(\left((x+2)^3\right)^{\frac{1}{3}}\right) + \frac{1}{2}\left[\ln\left(\frac{x}{(x^2+3x+2)^2}\right)\right]$

$= \ln(x+2) + \ln\left(\frac{x^{\frac{1}{2}}}{\left((x^2+3x+2)^2\right)^{1/2}}\right)$

$x^2+3x+2 = (x+1)(x+2)$

$= \ln\left[(x+2)\left(\frac{x^{\frac{1}{2}}}{x^2+3x+2}\right)\right] = \ln\left(\frac{x^{\frac{1}{2}}(x+2)}{(x+1)(x+2)}\right) = \boxed{\ln\left(\frac{x^{\frac{1}{2}}}{x+1}\right)}$

3–8 Find the exact value of each expression.

3. (a) $\log_5 125$ (b) $\log_3(\frac{1}{27})$

4. (a) $\ln(1/e)$ (b) $\log_{10}\sqrt{10}$

5. (a) $e^{\ln 4.5}$ (b) $\log_{10} 0.0001$

6. (a) $\log_{1.5} 2.25$ (b) $\log_5 4 - \log_5 500$

7. (a) $\log_2 6 - \log_2 15 + \log_2 20$
(b) $\log_3 100 - \log_3 18 - \log_3 50$

8. (a) $e^{-2\ln 5}$ (b) $\ln(\ln e^{e^{10}})$

$\log_5(5^3) = 3$

$\log_3(\frac{1}{27}) = \log_3(\frac{1}{3^3})$

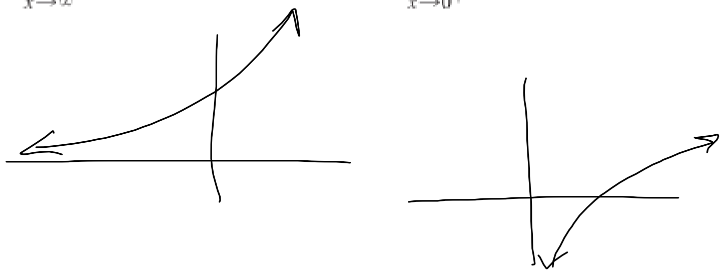
$= \log_3(3^{-3}) = -3$

$e^{\ln(5^{-2})} = 5^{-2}$

$= \frac{1}{25}$

$\boxed{4}$ If $a > 1$, then exponential growth

$$\lim_{x \to \infty} \log_a x = \infty \quad \text{and} \quad \lim_{x \to 0^+} \log_a x = -\infty$$



$$\log_e x = \ln x$$

5

$$\boxed{\ln x = y \iff e^y = x}$$

6

$$\boxed{\begin{array}{ll} \ln(e^x) = x & x \in \mathbb{R} \\ e^{\ln x} = x & x > 0 \end{array}}$$

This second piece of equation 6 is how we bootstrap to derivatives of exponential functions, whose base is NOT the natural base, *e*.

$$\boxed{\ln e = 1}$$

Duh.

7 **Change of Base Formula** For any positive number a ($a \neq 1$), we have

$$\log_a x = \frac{\ln x}{\ln a}$$

$\log_{10} x$
$\ln x$

All your calculator can do!

$y = \log_a x$

$a^y = a^{\log_a x}$

$a^y = x$

$\ln(a^y) = \ln(x)$

$y \ln(a) = \ln(x)$

$y = \dfrac{\ln(x)}{\ln(a)} = \log_a(x)$

8 $$\lim_{x \to \infty} \ln x = \infty \qquad \lim_{x \to 0^+} \ln x = -\infty$$

This is just a specific case of equation 4, since $e > 1$.

For future reference, an increasing power function will eventually out-grow an increasing log function. Here we're talking about bases for the log function that are greater than 1. If the base is between 0 and 1 then the log function is a decreasing function.

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$$

$$\frac{\ln x}{x^{.001}} \xrightarrow{x \to \infty} 0$$

These first couple questions are basically asking if you read the section. Look it up sort of deal.

1. (a) How is the logarithmic function $y = \log_a x$ defined?
(b) What is the domain of this function?
(c) What is the range of this function?
(d) Sketch the general shape of the graph of the function $y = \log_a x$ if $a > 1$.

2. (a) What is the natural logarithm?
(b) What is the common logarithm?
(c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

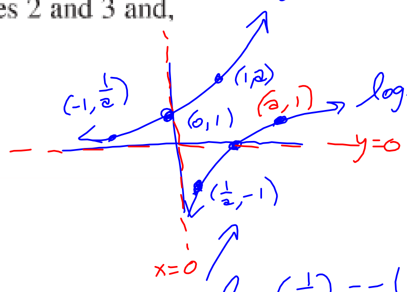
3

a^x $a > 1$ $3^x, 10^x$

23–24 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 2 and 3 and, if necessary, the transformations of Section 1.3.

23. (a) $y = \log_{10}(x + 5)$ (b) $y = -\ln x$

24. (a) $y = \ln(-x)$ (b) $y = \ln|x|$



$\log_a(x)$

$\log_3(x)$

$\log_{10}(x)$

$\log_a\left(\frac{1}{a}\right) = -1$

$\log_3\left(\frac{1}{3}\right) = -1$

$\log_3\left(3^{-1}\right) = -1$

$\log_3(.5x - 3) + 5$

$-5\log_3(2x + 4) - 2$

Graph $g(x) = \log_3\left(\frac{1}{2}x - 3\right) + 5$

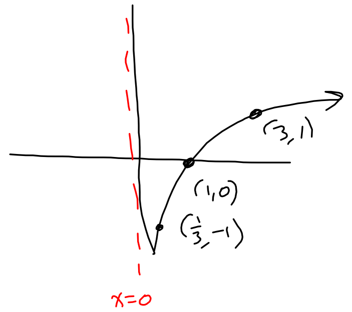
$= \log_3\left(\frac{1}{2}(x-6)\right) + 5$

1. $af(x)$
2. $f(2x)$
3. $f(x-c)$
4. $f(x)+d$

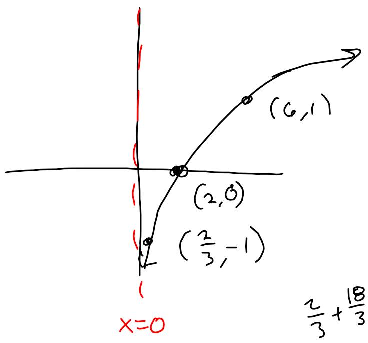
$\log_3(x) \to \log_3\left(\frac{1}{2}x\right) \to \log_3\left(\frac{1}{2}(x-6)\right)$

$\to \log_3\left(\frac{1}{2}(x-6)\right) + 5$

$f(x) = \log_3(x)$

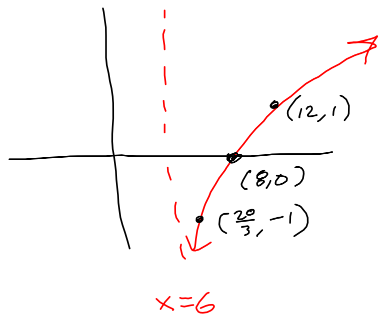


$f\left(\frac{1}{2}x\right) = \log_3\left(\frac{1}{2}x\right)$



$\frac{2}{3} + \frac{18}{3} = \frac{20}{3}$

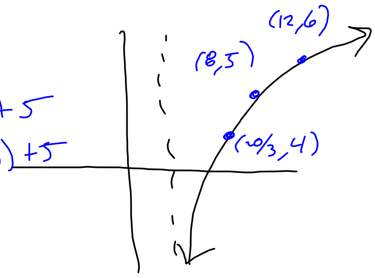
$f\left(\frac{1}{2}(x-6)\right) = \log_3\left(\frac{1}{2}(x-6)\right)$



$g(x) =$

$f\left(\frac{1}{2}(x-6)\right) + 5$

$= \log_3\left(\frac{1}{2}(x-6)\right) + 5$



25–26

(a) What are the domain and range of f?

(b) What is the x-intercept of the graph of f?

(c) Sketch the graph of f.

25. $f(x) = \ln x + 2$ **26.** $f(x) = \ln(x - 1) - 1$

$\ln(\boxed{})$ needs

$\boxed{} > 0$

These are pretty easy ones. Like to see you working with tougher ones...

27–36 Solve each equation for x.

27. (a) $e^{7-4x} = 6$ (b) $\ln(3x - 10) = 2$

28. (a) $\ln(x^2 - 1) = 3$ (b) $e^{2x} - 3e^x + 2 = 0$

29. (a) $2^{x-5} = 3$ (b) $\ln x + \ln(x - 1) = 1$

30. (a) $e^{3x+1} = k$ (b) $\log_2(mx) = c$

31. $e - e^{-2x} = 1$ **32.** $10(1 + e^{-x})^{-1} = 3$

33. $\ln(\ln x) = 1$ **34.** $e^{e^x} = 10$

35. $e^{2x} - e^x - 6 = 0$ **36.** $\ln(2x + 1) = 2 - \ln x$

$$e^{7-4x} = 6$$
$$\ln(e^{7-4x}) = \ln 6$$
$$7 - 4x = \ln 6$$
$$-4x = \ln 6 - 7$$
$$4x = 7 - \ln 6$$
$$\boxed{x = \frac{7 - \ln(6)}{4}}$$
$$\approx 1.302060133$$

35

$$(e^x)^2 - e^x - 6 = 0$$ Quadratic in Form

$$u = e^x$$
$$u^2 - u - 6 = 0$$
$$(u-3)(u+2) = 0$$
$$u = 3 \text{ or } u = -2 \rightarrow \cancel{\ }$$
$$e^x = 3$$
$$x = \ln 3$$

32

$$10(1 + e^{-x})^{-1} = 3$$
$$\frac{10}{e^{-x} + 1} = 3$$
$$10 = 3(e^{-x} + 1) = 3e^{-x} + 3$$
$$7 = 3e^{-x}$$
$$3e^{-x} = 7$$
$$e^{-x} = \frac{7}{3}$$
$$-x = \ln(7/3) \qquad -\ln(7/3)$$
$$x = -\ln(7/3) \qquad = \ln(3/7)$$
$$x = \ln\left(\frac{3}{7}\right)$$

37–38 Find the solution of the equation correct to four decimal places.

37. (a) $e^{2+5x} = 100$ (b) $\ln(e^x - 2) = 3$

38. (a) $\ln(1 + \sqrt{x}) = 2$ (b) $3^{1/(x-4)} = 7$

37 (a)

$$e^{5x+2} = 100$$

$$5x+2 = \ln(100)$$

$$5x = \ln(100) - 2$$

$$x = \frac{\ln(100)-2}{5} \approx 0.5210340372 \approx \boxed{.5210}$$

(b) $\ln(e^x - 2) = 3$

$$e^{\ln(e^x-2)} = e^3$$

$$e^x - 2 = e^3$$

$$e^x = e^3 + 2$$

$$\ln(e^x) = \ln(e^3+2)$$

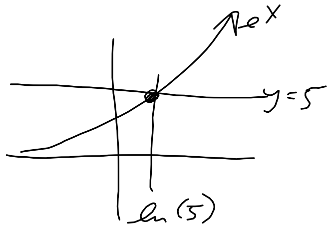
$$x = \ln(e^3+2)$$

calculator,

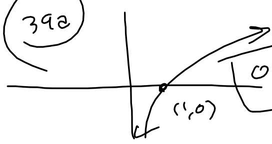
39–40 Solve each inequality for x.

39. (a) $\ln x < 0$ (b) $e^x > 5$

40. (a) $1 < e^{3x-1} < 2$ (b) $1 - 2\ln x < 3$



39a



$\boxed{0 < x < 1}$

$x \in (0,1)$

$\{x \mid 0 < x < 1\}$

39b

$e^x > 5$

$\ln x$ is increasing

$x > \ln(5)$

42. The velocity of a particle that moves in a straight line under the influence of viscous forces is $v(t) = ce^{-kt}$, where c and k are positive constants.
(a) Show that the acceleration is proportional to the velocity.
(b) Explain the significance of the number c.

$\frac{dy}{du} \cdot \frac{du}{dx}$

(a) $v(t) = ce^{-kt}$

$v'(t) = -kce^{-kt} = a(t)$

A is proportional to B means

$A = kB$ for some k.

$\frac{A}{B} = k$ for some k

$\frac{a(t)}{v(t)} = \frac{v'(t)}{v(t)} = \frac{-kce^{-kt}}{ce^{-kt}} = -k$

! Done. $-k$ is proportionality constant!

45. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is $n = f(t) = 100 \cdot 2^{t/3}$.
(a) Find the inverse of this function and explain its meaning.
(b) When will the population reach 50,000?

(b)

c = initial velocity

$Ce^{-k \cdot 0} = Ce^0 = C$

§6.3 II

47–52 Find the limit.

47. $\lim_{x\to 3^+} \ln(x^2 - 9)$

48. $\lim_{x\to 2^-} \log_5(8x - x^4)$

49. $\lim_{x\to 0} \ln(\cos x)$ $\to 1$

50. $\lim_{x\to 0^+} \ln(\sin x)$ $\to$ $\sin x \xrightarrow{x\to 0^+} 0^+$

51. $\lim_{x\to\infty} [\ln(1 + x^2) - \ln(1 + x)]$ $\to$ $\ln\left(\frac{x^2+1}{x+1}\right)$

52. $\lim_{x\to\infty} [\ln(2 + x) - \ln(1 + x)]$ $\to$ $\ln\left(\frac{x+2}{x+1}\right)$ Point is $\frac{x+2}{x+1} > 0$ is different from $x+2>0$ & $x+1>0$

$\lim_{x\to 2^-} \log_5(8x - x^4)$

Analyze $8x - x^4$

$= -x^4 + 8x$

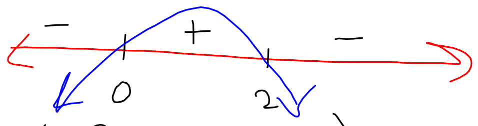
$= -x(x^3 - 8)$

$= -x(x-2)(x^2+2x+4)$

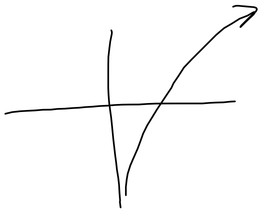
$x^3 - y^3 = (x-y)(x^2+xy+y^2)$

$x^3 = 8$ 1 real solution

$$\boxed{\lim_{x\to 2^-} \log_5(8x - x^4)} = \lim_{x\to 0^+} \log_5(x) \boxed{= -\infty}$$

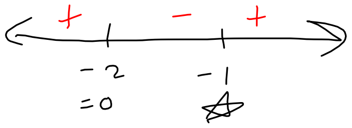


$\mathcal{D} = (0, 2)$

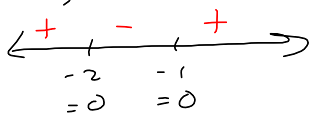


Need

$\frac{x+2}{x+1} > 0$



$(x+1)(x+2) > 0$

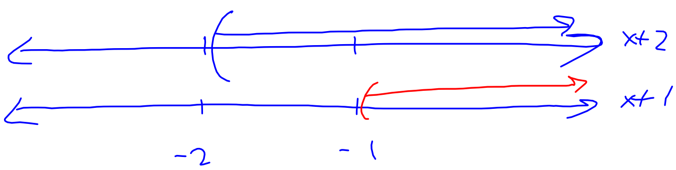


$\mathcal{D}\left(\ln\left(\frac{x+2}{x+1}\right)\right) = (-\infty, -2) \cup (-1, \infty)$

$\mathcal{D}(\ln(x+2) - \ln(x+1))$ is different!

Need

$x+2 > 0$ AND $x+1 > 0$



AND means intersect.

$\mathcal{D} = (-2, \infty) \cap (-1, \infty) = (-1, \infty)$

53–54 Find the domain of the function.

53. $f(x) = \log_{10}(x^2 - 9)$ **54.** $f(x) = \ln x + \ln(2 - x)$

55–57 Find (a) the domain of f and (b) f^{-1} and its domain.

55. $f(x) = \sqrt{3 - e^{2x}}$ **56.** $f(x) = \ln(2 + \ln x)$

57. $f(x) = \ln(e^x - 3)$

#54 brings up the point that how the original is written has an impact on the domain. If you combine the two terms in *f* , into one log, you get a bigger domain!

These domain questions are quite a bit meatier than the early ones.

$\sqrt{-e^{2x}+3} = f(x)$

$\mathcal{D} = \left(-\infty, \frac{\ln(3)}{2}\right]$

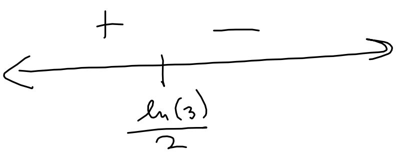
Need $-e^{2x}+3 \geq 0$

$-e^{2x} = -3$

$e^{2x} = 3$

$2x = \ln(3)$

$x = \frac{\ln(3)}{2}$



$\mathcal{D}$:

(1) $\frac{\text{stuff}}{0}$ Bad

Solve denominator $= 0$.
Throw solution out of $\mathcal{D}$

(2) $\sqrt{\text{Negative}}$ Bad

$\sqrt{\text{Radicand}}$

Need $\boxed{\text{radicand} \geq 0.}$ Solution set $= \mathcal{D}$.

Do NOT write

Need $\sqrt{\text{Radicand}} \geq 0$

58. (a) What are the values of $e^{\ln 300}$ and $\ln(e^{300})$?

(b) Use your calculator to evaluate $e^{\ln 300}$ and $\ln(e^{300})$. What do you notice? Can you explain why the calculator has trouble?

Calculator can choke on $e^{300} \approx 1.94 \times 10^{130}$

$$\left(8^{\frac{5}{3}}\right) = \left(8^5\right)^{\frac{1}{3}} = \left(\left(8\right)^{\frac{1}{3}}\right)^5$$

$$\left(32768\right)^{\frac{1}{3}} = 2^5 = 32$$

59–64 Find the inverse function.

59. $y = \ln(x + 3)$

60. $y = 2^{10^x}$

61. $f(x) = e^{x^3}$

62. $y = (\ln x)^2, \quad x \geqslant 1$

63. $y = \log_{10}\left(1 + \dfrac{1}{x}\right)$

64. $y = \dfrac{e^x}{1 + 2e^x}$

59

$x = \ln(y+3)$

$\ln(y+3) = x$

$e^{\ln(y+3)} = e^x$

$y+3 = e^x$

$y = \boxed{e^x - 3 = f^{-1}(x)}$

$y = \dfrac{e^x}{2e^x + 1}$

$\dfrac{e^y}{2e^y + 1} = x$

$e^y = 2xe^y + x$

$e^y - 2xe^y = x$

$e^y(1-2x) = x$

$e^y = \dfrac{x}{1-2x}$

$\boxed{y = \ln\left(\dfrac{x}{1-2x}\right) = f^{-1}(x)}$

65. On what interval is the function $f(x) = e^{3x} - e^{x}$ increasing?

66. On what interval is the curve $y = 2e^{x} - e^{-3x}$ concave downward?

Need $f''(x) > 0$ for #66.

65*

$e^{5x} - e^{2x} = f(x)$

$f'(x) = 5e^{5x} - 2e^{2x} \overset{\text{SET}}{=} 0$

$e^{2x}(5e^{3x} - 2) = 0$

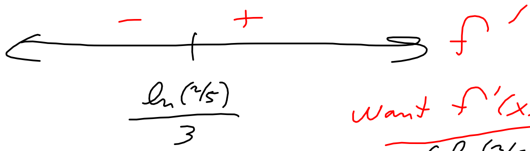
$5e^{3x} - 2 = 0$

$5e^{3x} = 2$

$e^{3x} = \frac{2}{5}$

$3x = \ln(2/5)$

$x = \frac{\ln(2/5)}{3}$



want $f'(x) > 0$

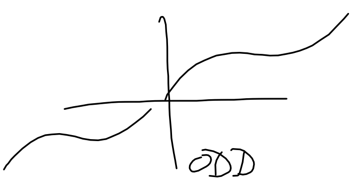
So $\boxed{\left(\frac{\ln(2/5)}{3}, \infty\right)}$

67. (a) Show that the function $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ is an odd function.

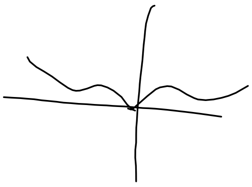
(b) Find the inverse function of f.

68. Find an equation of the tangent to the curve $y = e^{-x}$ that is perpendicular to the line $2x - y = 8$.

69. Show that the equation $x^{1/\ln x} = 2$ has no solution. What can you say about the function $f(x) = x^{1/\ln x}$?



ODD



$f(-x) = -f(x)$

$\boxed{\text{68 ish}}$

$y = e^{5x}$

perp. to $3x + 5y = 7$

$5y = -3x + 7$

$y = -\frac{3}{5}x + \frac{7}{5} \Rightarrow m = -\frac{3}{5}$

$\Longrightarrow m_{\perp} = \frac{5}{3} = m_{tan}$

$y' = 5e^{5x} \overset{SET}{=} m_{tan} = \frac{5}{3}$

$e^{5x} = \frac{1}{3}$

$5x = \ln(1/3) = \ln(1) - \ln(3)$

$= 0 - \ln(3)$

$5x = -\ln(3)$

$\boxed{x_1 = -\frac{\ln(3)}{5}}$

$y_1 = e^{5\left(-\frac{\ln(3)}{5}\right)}$

$= e^{-\ln(3)}$

$= e^{\ln\left(\frac{1}{3}\right)}$

$= \boxed{\frac{1}{3} = y_1}$

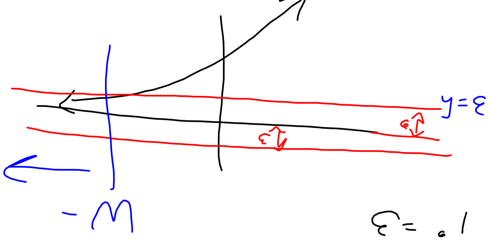
$y = m(x - x_1) + y_1$

$y = \frac{5}{3}\left(x + \frac{\ln(3)}{5}\right) + \frac{1}{3}$

71. Let $a > 1$. Prove, using Definitions 3.4.6 and 3.4.7, that
(a) $\lim_{x \to -\infty} a^x = 0$ (b) $\lim_{x \to \infty} a^x = \infty$

This #71 is like a 300-level question.

$\lim_{x \to -\infty} a^x = 0$ means given $\varepsilon > 0$, $\exists\, M > 0 \ni$
$|a^x - 0| < \varepsilon$ whenever $x < -M$



Assume $a > 1$

$\varepsilon = .1$

want $|a^x - 0| < .1$

$a^x < .1$

$x < \log_a(.1) < 0$

Let $M = -\log_a(.1)$

$|a^x - 0| < \varepsilon$

$a^x < \varepsilon$

$x < \log_a(\varepsilon)$

Let $\varepsilon > 0$ Define $M = -\log_a(\varepsilon)$

Then $x < -M$, we have

$|a^x - 0| = a^x < a^{\log_a(\varepsilon)} = \varepsilon$ $\blacksquare$