6.2 Exponential Functions and Their Derivatives

The first few pages of this section are a lot of hand-waving, to convince you that exponential functions are continuous. I don't think Calc II students are generally sophisticated enough to really need this, let alone even question whether these functions behave as advertised. So first few pages are kind of fluff, that you aren't going to care about, much, until/unless you're a math major, long after Calculus II.

$$f(x) = a^x$$ $3^x, 2^x, (\frac{1}{3})^x$

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}} \qquad a^{-n} = \frac{1}{a^n}$$

$$a^x = a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$$

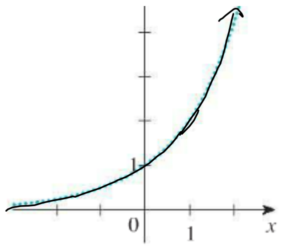


FIGURE 1
Representation of $y = 2^x$, x rational

$2^{\sqrt{3}}$?

$2^{1.73} = 2^{\frac{173}{100}} = \sqrt[100]{2^{173}}$

$2^{1.73} < 2^{\sqrt{3}} < 2^{1.74}$

$$1.73 < \sqrt{3} < 1.74 \quad \Rightarrow \quad 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74}$$
$$1.732 < \sqrt{3} < 1.733 \quad \Rightarrow \quad 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733}$$
$$1.7320 < \sqrt{3} < 1.7321 \quad \Rightarrow \quad 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321}$$
$$.73205 < \sqrt{3} < 1.73206 \quad \Rightarrow \quad 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206}$$

$$2^{\sqrt{3}} \approx 3.321997$$

1 $$\boxed{a^x = \lim_{r \to x} a^r \qquad r \text{ rational}}$$

Summarizing more precalculus notions:

2 Theorem If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. In particular, $a^x > 0$ for all x. If $0 < a < 1$, $f(x) = a^x$ is a decreasing function; if $a > 1$, f is an increasing function. If $a, b > 0$ and $x, y \in \mathbb{R}$, then

1. $a^{x+y} = a^x a^y$ **2.** $a^{x-y} = \dfrac{a^x}{a^y}$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$

$2^{2+3} = 2^5 = 2^2 2^3$

$3^{2x+5} = 3^{2x} \cdot 3^5$

$= (3^5) \cdot 3^{2x}$

$= (3^5)(3^2)^x$ OR $(3^5)(3^{x \cdot 2})$

$= 3^5 \cdot 9^x$ $= (3^5)(3^x)^2$

$\dfrac{3^2}{3^1} = 3^{2-1}$

$(3x)^3 = 3^3 x^3 = 27x^3$

$(3+x)^3 = 3^3 + x^3$

Nooooo!

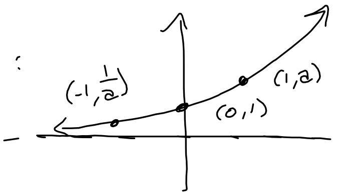
See Binomial Theorem!

3 If $a > 1$, then $\lim_{x\to\infty} a^x = \infty$ and $\lim_{x\to-\infty} a^x = 0$

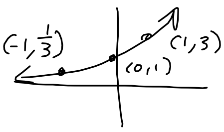
If $0 < a < 1$, then $\lim_{x\to\infty} a^x = 0$ and $\lim_{x\to-\infty} a^x = \infty$

$\Big\}$ $a^x > 0$ always

$a > 1$:

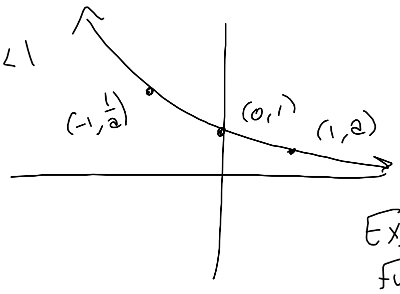


Exponential Growth

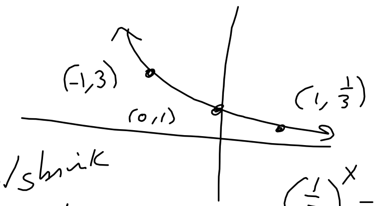


$3^{-10000} = \frac{1}{3^{10000}} > 0$, although close to 0.

$0 < a < 1$



Exponential Decay



Exponential functions grow/shrink in proportion to their size

$\left(\frac{1}{3}\right)^x = f(x)$

$y' = ky$ (for some k)

Preview: e^x is where $k = 1$

Finally! Some Calculus!!! We stumble over the natural base for exponential functions, *e* .
Euler named it after himself, because he was this great man and stuff.

We characterize the number *e* in a number of ways, depending on the context. For our purposes, we're looking for a 'natural base' so that the corresponding derivative comes out as cleanly as possible.

7 Definition of the Number *e*

$$e \text{ is the number such that } \lim_{h\to 0}\frac{e^h-1}{h}=1$$

$e \approx 2.718281828$ and no, it's not repeating

$$f(x)=a^x \Longrightarrow \frac{f(x+h)-f(x)}{h} = \frac{a^{x+h}-a^x}{h} = \frac{a^x a^h-a^x}{h} = \frac{a^x(a^h-1)}{h}$$

$$=\left(\frac{a^h-1}{a^h}\right)a^x \xrightarrow{h\to 0}$$

Note if we're looking for $f'(0)$, we have

$$\lim_{h\to 0}\left(\frac{a^h-1}{a^h}\right)a^0 = \lim_{h\to 0}\frac{a^h-1}{h} = f'(0) \text{ for this thing}$$

up-shot is that $\frac{d}{dx}\left[a^x\right] = f'(0)\,a^x$

Let $e=a$ with the property that $\lim_{h\to 0}\frac{e^h-1}{h}=1$

$$\frac{e^{x+h}-e^x}{h} = \left(\frac{e^h-1}{h}\right)e^x \xrightarrow{h\to 0} 1\cdot e^x = e^x$$

Punchline $\frac{d}{dx}\left[e^x\right] = e^x = y = y'$

$y' = ky \qquad k=1$

8 **Derivative of the Natural Exponential Function**

$$\frac{d}{dx}(e^x) = e^x$$

Chain Rule applied to exponential function.

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$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

$\frac{d}{dx}\left[e^{x^2}\right]$

$=(e^{x^2})(2x)$

$=2xe^{x^2}$

e^x lines between 2^x & 3^x

10 **Properties of the Natural Exponential Function** The exponential function $f(x) = e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus $e^x > 0$ for all x. Also

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$$

So the x-axis is a horizontal asymptote of $f(x) = e^x$.

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$$\boxed{\int e^x \, dx = e^x + C}$$

EXAMPLES FROM HOMEWORK EXERCISES

S6.2

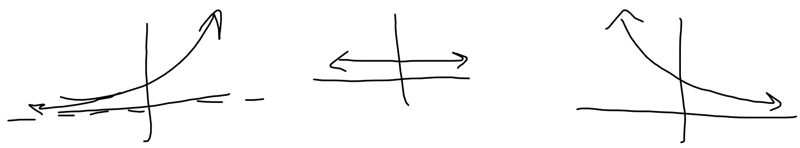
1. (a) Write an equation that defines the exponential function with base $a > 0$. $f(x) = a^x$

(b) What is the domain of this function? $\mathcal{D}(a^x) = \mathbb{R} = (-\infty, \infty)$

(c) If $a \neq 1$, what is the range of this function? $\mathcal{R}(a^x) = (0, \infty)$

(d) Sketch the general shape of the graph of the exponential function for each of the following cases.

(i) $a > 1$ (ii) $a = 1$ (iii) $0 < a < 1$



2. (a) How is the number e defined?
(b) What is an approximate value for e?
(c) What is the natural exponential function?

(a) e is the real number satisfying

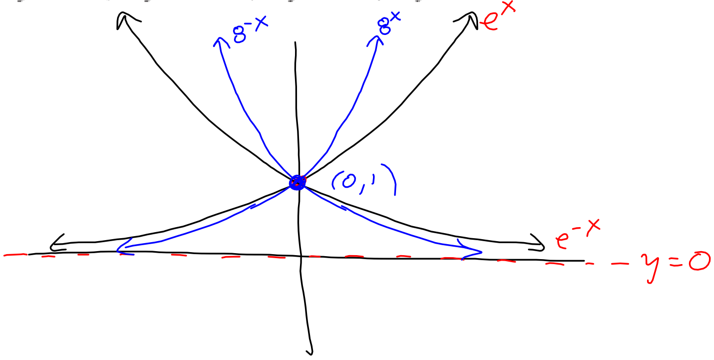
$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

(b) $2.718281828 \approx e$

(c) $y = e^x$ is natural exponential function.

3–6 Graph the given functions on a common screen. How are these graphs related?

4. $y = e^x$, $y = e^{-x}$, $y = 8^x$, $y = 8^{-x}$



REVIEW OF GRAPHING BY TRANSFORMING BASIC FUNCSIONS

7–12 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graphs given in Figures 3 and 12 and, if necessary, the transformations of Section 1.3.

$f(x)$

Makes sense, intuitively — Vertical stretch (shrink) — $af(x)$ — $(x,y) \longmapsto (x, ay)$ — Includes reflection

multiply y-values by a.

Opposite what you think — Horizontal stretch (shrink) — $f(bx)$ — $(x,y) \longmapsto (\frac{1}{b}x, y)$ — Includes reflection

Divide x-values by b.

$f(x) = \sqrt{x}$

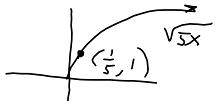
$f(5x) = \sqrt{5x}$

Horizontal Shift — $f(x+c)$ — $(x,y) \longmapsto (x-c, y)$

SUBTRACT c from x-values

Makes sense, intuitively — Vertical Shift — $f(x)+d$ — $(x,y) \longmapsto (x, y+d)$

Add d to y-values



Example $g(x) = -3\sqrt{10-5x} + 7$

Want to "see" $g(x)$ as a sequence of transformations on basic function, $f(x) = \sqrt{x}$. Want to re-write function in the form $af(b(x+c)) + d$, to break it down into steps.

1st "trick" $10 - 5x = -5x + 10 = -5(x-2)$, so

$g(x) = -3\sqrt{10-5x} + 7 = -3\sqrt{-5(x-2)} + 7$

Sequence of graphs:

1. $f(x) = \sqrt{x}$
2. $-3f(x) = -3\sqrt{x}$
3. $-3f(-5x) = -3\sqrt{-5x}$
4. $-3f(-5(x-2)) = -3\sqrt{-5(x-2)}$
5. $-3f(-5(x-2)) + 7 = -3\sqrt{-5(x-2)} + 7 = g(x)$

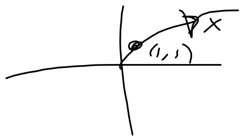
Track where $(1,1)$ goes

1. $(1,1)$
2. $(1,-3)$
3. $(-\frac{1}{5},-3)$
4. $(9/5, -3)$
5. $(\frac{9}{5}, 4)$

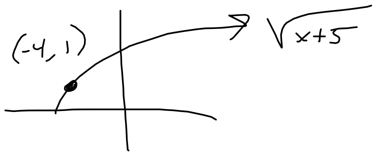
$-\frac{1}{5}+2 = \frac{-1+10}{5} = \frac{9}{5}$

$-3+7=4$

$f(x) = \sqrt{x}$



$f(x+5) = \sqrt{x+5}$



$(1,1) \longmapsto (-4,1)$

Rewrite $ax+b$ inside as $a\left(x+\frac{b}{a}\right)$

$10-5x = -5x+10 = -5(x-2)$

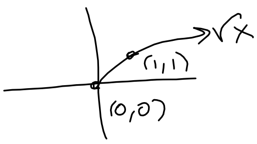
$g(x) = -3\sqrt{10-5x} + 7$

$= -3\sqrt{-5x+10} + 7$

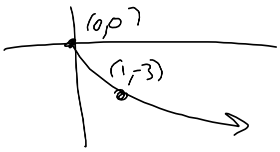
$= -3\sqrt{-5(x-2)} + 7$

① $\sqrt{x}$ ② $-3\sqrt{x}$ Flip & stretch vert. ③ $-3\sqrt{-5x}$ Flip & shrink horizont. ④ $-3\sqrt{-5(x-2)}$ Right 2 ⑤ $-3\sqrt{-5(x-2)}+7$ up 7

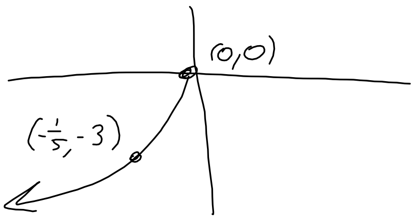
$\sqrt{x} = f(x)$



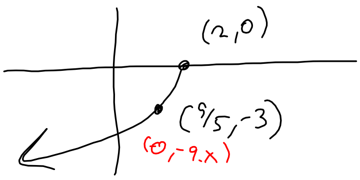
$-3\sqrt{x} = -3f(x)$



$-3\sqrt{-5x}$



$-3\sqrt{-5(x-2)}$ Delay

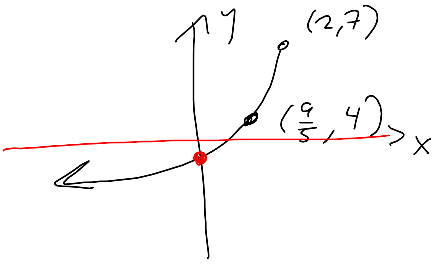


$-\frac{1}{5}+2 = -\frac{1}{5}+\frac{10}{5} = \frac{9}{5}$

$= -3\sqrt{10}$

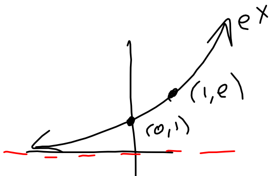
Below -9

$-3\sqrt{-5(x-2)} + 7$

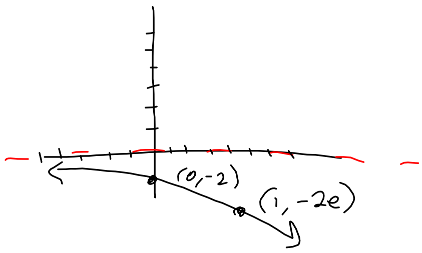


12. $y = 2(1 - e^x) = 2 - 2e^x = -2e^x + 2$

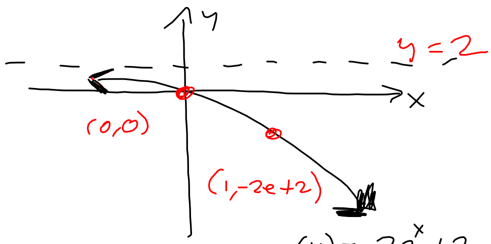
$e^x \longrightarrow -2e^x \longrightarrow -2e^x + 2$



$f(x) = e^x$



$-2f(x) = -2e^x$



$g(x) = -2e^x + 2$

15–16 Find the domain of each function.

16. (a) $g(t) = \sin(e^{-t})$ (b) $g(t) = \sqrt{1 - 2^t}$

$\mathcal{D}(\sin(u)) = \mathbb{R}$

$\mathcal{D}(e^{-t}) = \mathbb{R}$

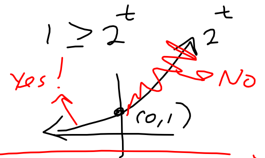
$f(t) = \sin(t)$

$g(t) = e^{-t}$

$$\mathcal{D}(f \circ g) = \{t \mid t \in \mathcal{D}(g) \text{ and } g(t) \in \mathcal{D}(f)\}$$
$$= \{t \mid t \in \mathcal{D}(e^{-t}) \text{ and } e^{-t} \in \mathcal{D}(\text{sine})\}$$
$$= \{t \mid t \text{ is real and } e^{-t} \text{ is real}\}$$
$$= \mathbb{R}$$

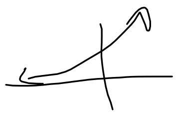
$\mathcal{D}$: Need $1 - 2^t \geq 0$

$1 \geq 2^t$



$\boxed{\mathcal{D}(g) = (-\infty, 0]}$

23–30 Find the limit.



$1.0001 > 1$

23. $\lim_{x\to\infty} (1.001)^x = \infty$

$$\frac{e^{3x}}{e^{3x}} \qquad \frac{e^{-3x}}{e^{3x}} = e^{-3x-3x} = e^{-6x}$$

25. $\lim_{x\to\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

$$\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3y}} = \frac{e^{3x}\left(1-e^{-6x}\right)}{e^{3x}\left(1+e^{-6x}\right)} = \frac{1-e^{-6x}}{1+e^{-6x}} \xrightarrow{x\to\infty} \frac{1}{1} = 1$$

$e^{-6x} \to 0$

31–50 Differentiate the function.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$(fg)' = f'g + fg'$$

$$e^x e^x = e^{x+x} = e^{2x}$$

34. $y = \dfrac{e^x}{1 - e^x}$

36. $y = e^{-2t}\cos(4t)$

34 $$y' = \frac{e^x(1-e^x) - e^x(-e^x)}{(1-e^x)^2} = \frac{e^x - e^{2x} + e^{2x}}{(\quad)^2} = \frac{e^x}{(1-e^x)^2}$$

36 $$-2e^{-2t}\cos(4t) + e^{-2t}(-4\sin(4t)) = y'$$

$$= -2e^{-2t}\cos(4t) - 4e^{-2t}\sin(4t)$$

46. $y = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$ Assume $y = f(u)$

u is the variable with respect to which we're differentiating

48. $y = \sqrt{1 + xe^{-2x}}$ $= (1 + xe^{-2x})^{\frac{1}{2}}$

46 $y' = \dfrac{(e^u + e^u)(e^u + e^{-u}) - (e^u - e^{-u})(e^u - e^{-u})}{(e^u + e^{-u})^2}$

51–52 Find an equation of the tangent line to the curve at the given point.

$y = m(x-x_1) + y_1$ Point-Slope

52. $y = \dfrac{e^x}{x}$, $(1, e) = (x_1, y_1)$

$y = f'(x_1)(x-x_1) + f(x_1)$

Need $m_{tan} = m = f'(x_1) = f'(1)$.

$y' = \dfrac{e^x \cdot x - e^x \cdot 1}{x^2} = \dfrac{xe^x - e^x}{x^2} = f'(x)$

Horizontal Line!

$y'(x_1) = y'(1) = f'(x_1) = f'(1) = \dfrac{1e^1 - e^1}{1^2} = \dfrac{0}{1} = 0 = m = f'(1)$

$y = 0(x-1) + e$

$\boxed{y = e}$

53. Find y' if $e^{x/y} = x - y$.

Implicit Differentiation.

Assume x is independent variable, since $y' = \frac{dy}{dx}$

$$e^{\frac{x}{y}} = x - y$$

$$\frac{d}{dx}\left[e^{u(x)}\right] = e^{u(x)} \cdot u'(x) = u'(x)e^{u(x)}$$

$$u(x) = \frac{x}{y} \Rightarrow u'(x) = \frac{1 \cdot y - x \cdot y'}{y^2}$$

$$\left(\frac{y - xy'}{y^2}\right)e^{\frac{x}{y}} = 1 - y'$$

$$(y - xy')e^{x/y} = (1 - y')y^2$$

$$ye^{\frac{x}{y}} - xy'e^{\frac{x}{y}} = y^2 - y^2y'$$

$$-xy'e^{\frac{x}{y}} + y^2y' = y^2 - ye^{\frac{x}{y}}$$

$$y'\left(-xe^{\frac{x}{y}} + y^2\right) = y^2 - ye^{\frac{x}{y}}$$

$$\boxed{y' = \frac{y^2 - ye^{\frac{x}{y}}}{y^2 - xe^{x/y}}}$$

OPTIONAL PORTION OF THE PRESENTATION:
ANOTHER DERIVATION OF THE NUMBER *e*.

In precalculus and mathematics of finance, we stumble across the natural base, in a wholly different way, which is equivalent, but we're not going to go into much detail on.

Compound Interest:
Let r = the annual percentage rate
m = the number of periods per year for compounding purposes
t = the number of years the money sits
P = the principal amount
A = the compound amount, after time t.

Then $A = P\left(1+\frac{r}{m}\right)^{mt}$

Continuous Compounding

FACT The more periods/yr, the closer this comes to Pe^{rt} !

"Proof" $\lim_{x\to\infty}(1+x)^{\frac{1}{x}} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$!