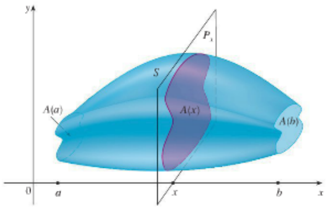
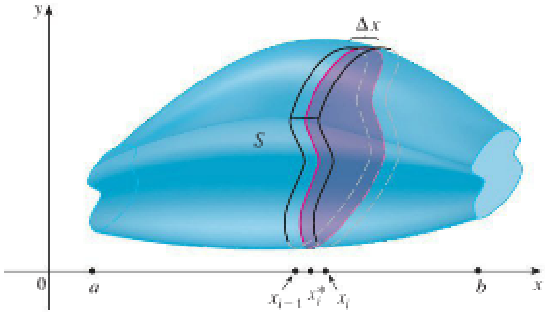
Calculus I
Section 5.2 - Volumes of Solids of Revolution: Disc (Washer) and General Methods

The General Idea



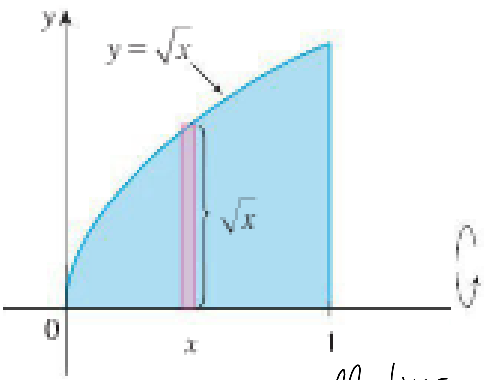


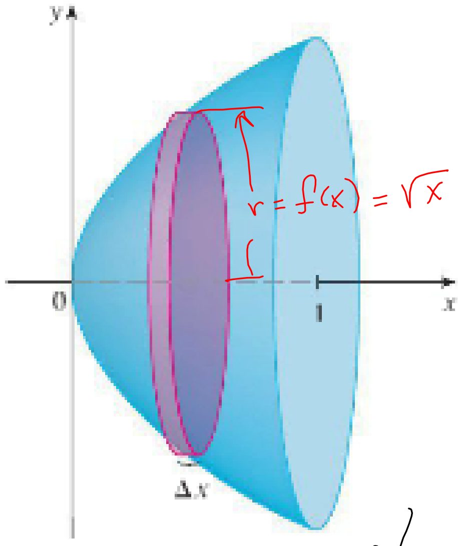
Area times thickness

$$V_i \approx A(x_i)\,\Delta x$$

$$V \approx \sum_{i=1}^{n} A(x_i)\,\Delta x$$

$$V \approx \sum_{i=1}^{n} A(x_i)\,\Delta x \xrightarrow{n \to \infty} \int_a^b A(x_i)\,\Delta x$$





Cross-secs are all discs for the solid of revolution

$$A(x_i) = \pi r^2 = \pi \left(f(x_i)\right)^2$$

$$V(x_i) \approx \pi f(x_i)^2 \Delta x$$

$$V \approx \sum_{i=1}^{n} \pi f(x_i)^2 \Delta x \xrightarrow{n \to \infty} \pi \int_a^b f(x)^2\,dx = V$$

Equal!

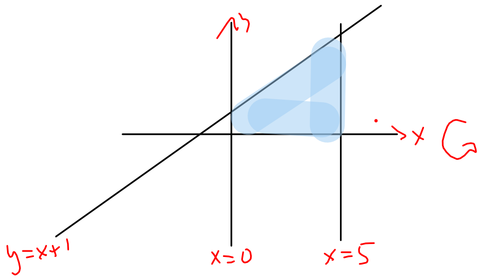
1. Question Details SCalc8 5.2.001. [3395019]

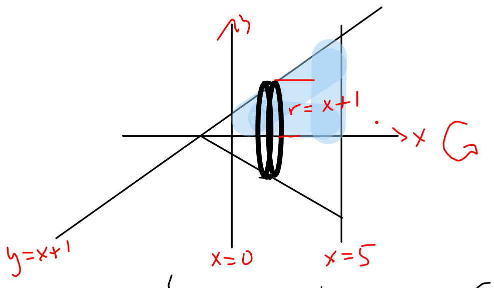
Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line.

$y = x + 1,\ y = 0,\ x = 0,\ x = 5;$ about the x-axis

Sketch the region.

Sketch the solid, and a typical disk or washer.

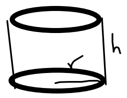




$$V = \pi \int_0^1 r^2\,dx = \pi \int_0^1 f(x)^2\,dx = \pi \int_0^1 (x+1)^2\,dx$$

V= area times thickness

$= \pi f(x)^2 \cdot \Delta x$



$v = \pi r^2 h$ right circular cylinder.

$$= \pi \int_0^1 (x^2+2x+1)\,dx = \pi \left[\tfrac{1}{3}x^3 + x^2 + x\right]_0^1 = \left(\tfrac{1}{3}+1+1\right) \pi = \boxed{\tfrac{7}{3}\pi} = V.$$

$$\int_0^5 = \pi\left[\tfrac{1}{3}(5)^3 + 5^2 + 5\right] = \left[\tfrac{125}{3} + \tfrac{75}{3} + \tfrac{15}{3}\right]\pi = \boxed{\tfrac{215\pi}{3}}$$

2. Question Details SCalc8 5.2.003. [3353852]

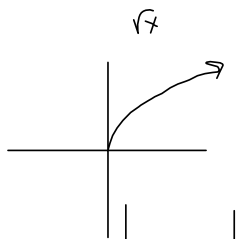
Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line.

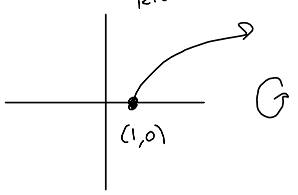
$y = \sqrt{x - 1},\ y = 0,\ x = 4;$ about the x-axis

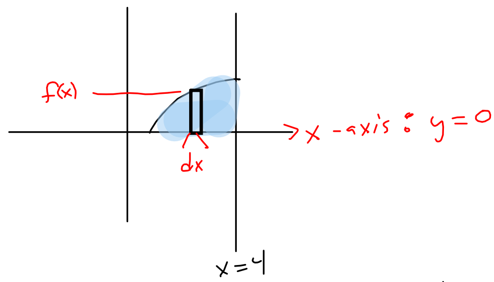
Sketch the region.

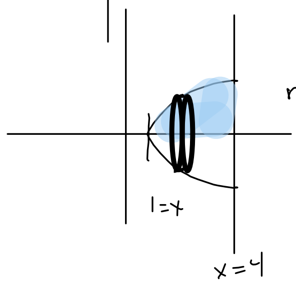
Sketch the solid, and a typical disk or washer.

$\sqrt{x-1}$ RIGHT limit (Delay)









$r = \sqrt{x-1}$

thickness $= dx$

$$V = \pi \int_1^4 \left(\sqrt{x-1}\right)^2 dx = \pi \int_1^4 (x-1)\,dx = \pi \left[\tfrac{1}{2}x^2 - x\right]_1^4$$

$$= \pi\left[8 - 4 - \left(\tfrac{1}{2} - 1\right)\right] = \pi\left[4 - \left(-\tfrac{1}{2}\right)\right] = \boxed{\tfrac{9\pi}{2} = V}$$

3. Question Details SCalc8 5.2.005. [3353736]

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line.

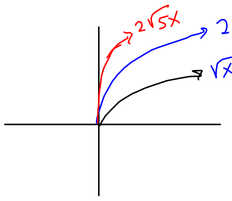
$x = 2\sqrt{5y}$, $x = 0$, $y = 3$; about the y-axis

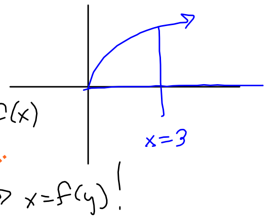
Sketch the region.

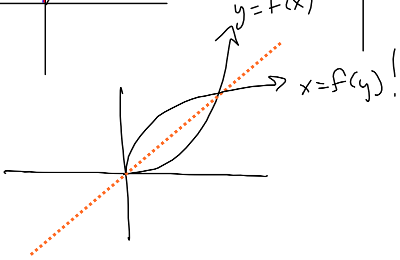
Sketch the solid, and a typical disk or washer.

$x = g(y)$: when in doubt, switch variables $y = g(x)$. Then switch back.

$y = 2\sqrt{5x}$, $y = 0$, $x = 3$







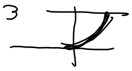
ORIGINAL :

$x = 2\sqrt{5y} \rightarrow \frac{x}{2} = \sqrt{5y}$

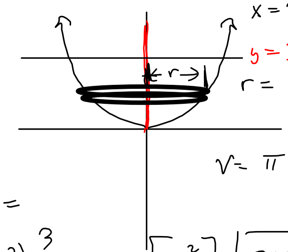
$\frac{x^2}{4} = 5y$

$y = \frac{x^2}{20}$

$y = 3$



Swapping Variables : Reflect about $y = x$



$x = 2\sqrt{5y}$

$y = 3$

$r = 2\sqrt{5y}$

$V = \pi \int_0^3 (2\sqrt{5y})^2 dy$

$= \pi \int_0^3 4(5y)\,dy = 20\pi \int_0^3 y\,dy =$

$20\pi \left(\frac{1}{2}y^2\right)\Big|_0^3 = 10\pi\left[3^2\right] = \boxed{90\pi}$

4. Question Details SCalc8 5.2.007.MI. [3353728]

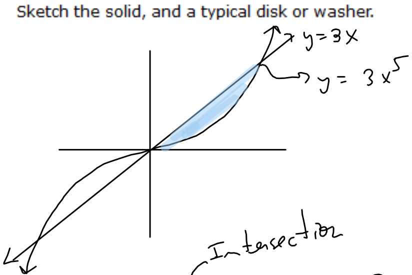
Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line.

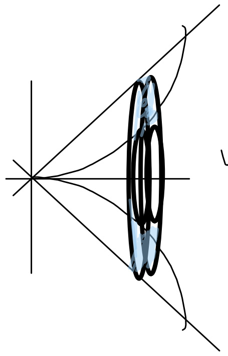
$y = 3x^5$, $y = 3x$, $x \ge 0$; about the x-axis

Sketch the region.

Sketch the solid, and a typical disk or washer.

A washer is disc with a hole in the middle.





Outer − Inner to get at the washer.

$V = \pi \int \left((3x)^2 - (3x^5)^2\right) dx$

volume for $y = 3x$

volume for $y = 3x^5$

Intersection

$\pi \int_0 \sim$

$3x = 3x^5$

$3x^5 - 3x = 0$

$3x(x^4 - 3) = 0$

$(x^2 - \sqrt{3})(x^2 + \sqrt{3})$

$(x - \sqrt[4]{3})(x + \sqrt[4]{3})(x^2 + \sqrt{3})$

$x = \sqrt[4]{3}$

So $\pi \int_0^{\sqrt[4]{3}} (9x^2 - 9x^{10})\,dx = V$

$= \pi \left[3x^3 - \frac{9}{11}x^{11}\right]_0^{\sqrt[4]{3}} = \pi\left[3(\sqrt[4]{3})^3 - \frac{9}{11}(\sqrt[4]{3})^{11}\right]$

$(3^{\frac{1}{4}})^3 = 3^{\frac{3}{4}}$ $\quad (3^{\frac{1}{4}})^{11} = 3^{\frac{11}{4}}$

$= 3^{\frac{8}{4}+\frac{3}{4}} = 3^2 \cdot 3^{\frac{3}{4}} = 9 \cdot 3^{\frac{3}{4}}$

$= \pi \cdot 3^{\frac{3}{4}} - \frac{9}{11} \cdot 9 \cdot 3^{\frac{3}{4}}$

$= \boxed{3^{\frac{3}{4}}\left[\pi - \frac{81}{11}\right]}$

Nov 17-7:56 AM

5. Question Details SCalc8 5.2.009. [3353938]

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line.

$y^2 = 2x, x = 2y$; about the y-axis

Sketch the region.

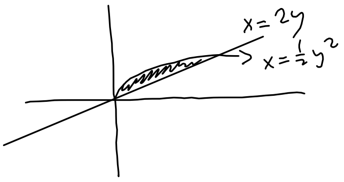
Sketch the solid, and a typical disk or washer.

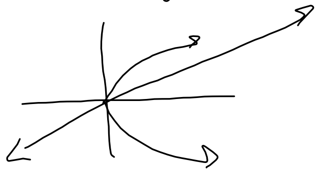
$x = \frac{1}{2}y^2 \Rightarrow y^2 = 2x$

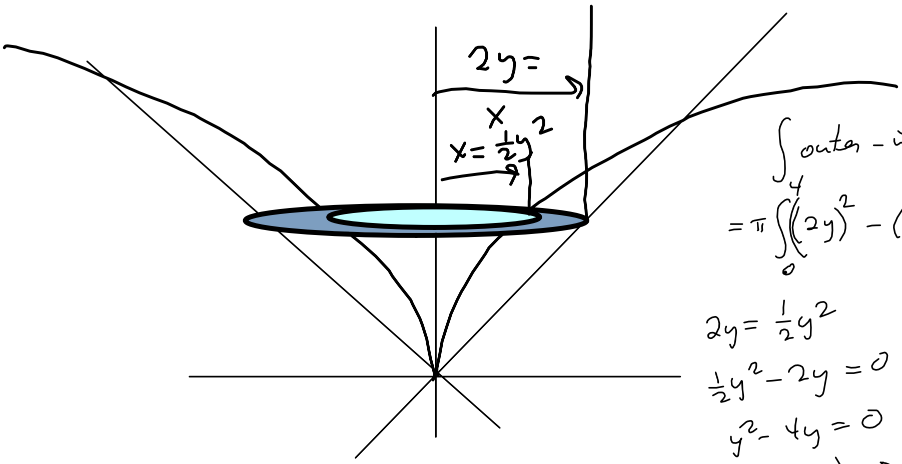
$\Rightarrow y = \pm\sqrt{2x}$

$x = 2y$

$\Rightarrow y = \frac{1}{2}x$







$\int_0^4$ outer − inner

$= \pi\int_0^4 \left((2y)^2 - (\sqrt{2y})^2\right)dy$

$2y = \frac{1}{2}y^2$

$\frac{1}{2}y^2 - 2y = 0$

$y^2 - 4y = 0$

$y(y-4) = 0 \Rightarrow y = 4$ is upper.

$\hookrightarrow y = 0$ is lower limit

$$= \pi\int_0^4 \left((2y)^2 - (\sqrt{2y})^2\right)dy$$

$$= \pi\int_0^4 (4y^2 - 2y)\,dy = \pi\left[\frac{4}{3}y^3 - y^2\right]_0^4 = \pi\left[\frac{4}{3}(64) - 16 - (0-0)\right]$$

$$= \pi\left[\frac{256}{3} - \frac{48}{3}\right] = \pi\left[\frac{208}{3}\right] = \boxed{\frac{208\pi}{3}}$$

Nov 17-7:57 AM

6. Question Details SCalc8 5.2.011. [3354078]

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line.

$y = x^2$, $x = y^2$; about $y = 1$

Sketch the region.

Sketch the solid, and a typical disk or washer.

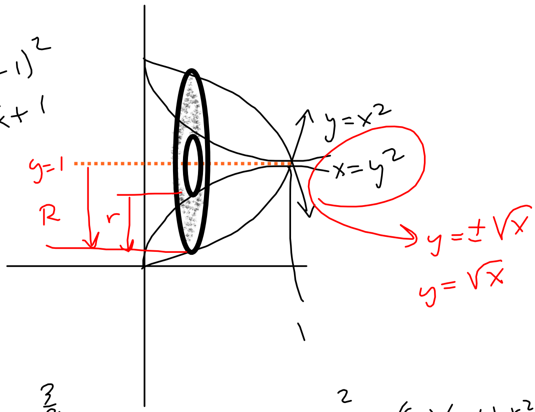
$R = 1 - x^2$

$r = 1 - \sqrt{x}$

$dx =$ increment

$(1-\sqrt{x})^2 = (\sqrt{x}-1)^2$

$= x - 2\sqrt{x} + 1$



$$\pi\int_0^1 \left((1-x^2)^2 - (1-\sqrt{x})^2\right)dx$$

$$= \pi\int_0^1 \left(x^4 - 2x^2 + 1 - (x - 2x^{\frac{1}{2}} + 1)\right)dx$$

$$2 \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{4}{3}x^{\frac{3}{2}}$$

$$(1-x^2)^2 = \left((-1)(-1+x^2)\right)^2 = (-1)^2(-1+x^2)^2 = (x^2-1)^2$$

$$= \pi\int_0^1 \left(x^4 - 2x^2 - x + 2x^{\frac{1}{2}}\right)dx$$

$$= \pi\left[\frac{1}{5}x^5 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}}\right]_0^1$$

$$= \pi\left[\frac{1}{5} - \frac{2}{3} - \frac{1}{2} + \frac{4}{3}\right] = \pi\left[\frac{1}{5} - \frac{1}{2} + \frac{2}{3}\right] = \pi\left[\frac{6-15+20}{30}\right] = \boxed{\frac{11\pi}{30}}$$

7. Question Details SCalc8 5.2.013. [3353716

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line

$y = 4 + \sec(x)$, $\frac{-\pi}{3} \le x \le \frac{\pi}{3}$, $y = 6$; about $y = 4$

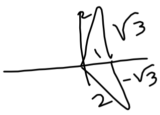
Sketch the region.

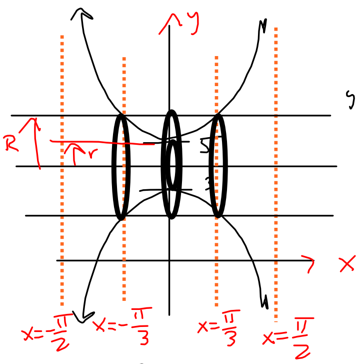
Sketch the solid, and a typical disk or washer.

$4 + \sec x = 6$

$\sec x = 2$

$\cos x = \frac{1}{2}$





$R = 6 - 4 = 2$

$r = (4 + \sec(x) - 4)$

$= \sec(x)$

$$\pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2^2 - \sec^2 x)\,dx$$

$$= 2\pi \int_0^{\frac{\pi}{3}} (4 - \sec^2 x)\,dx = 2\pi \left[4x - \tan x\right]_0^{\frac{\pi}{3}}$$

$$= 2\pi \left[\frac{4\pi}{3} - \tan\frac{\pi}{3}\right] = \boxed{\frac{8\pi^2}{3} - 2\pi\sqrt{3}}$$

Nov 17-7:57 AM

8. Question Details SCalc8 5.2.014. [3353818]

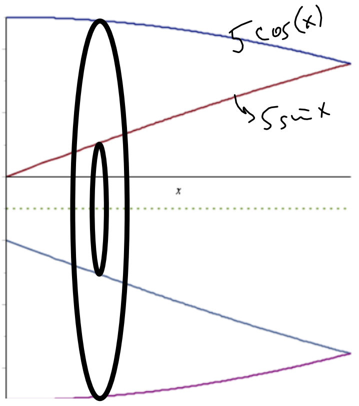
Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line.

$y = 5\sin(x), \quad y = 5\cos(x), \quad 0 \le x \le \pi/4; \quad \text{about } y = -1$

$\frac{5}{\sqrt{2}}$

Sketch the region.

Sketch the solid, and a typical disk or washer.



$$\pi \int_0^{\frac{\pi}{4}} \left((5\cos(x))^2 - (5\sin(x))^2 \right) dx$$

$$= 25\pi \int_0^{\frac{\pi}{4}} \left(\cos^2(x) - \sin^2(x)\right) dx$$

$$= 25\pi \int_0^{\frac{\pi}{4}} \cos(2x)\, dx$$

$u = 2x$

$du = 2dx$

$$= \frac{25\pi}{2} \int_0^{\frac{\pi}{4}} \cos(2x)\,(2dx)$$

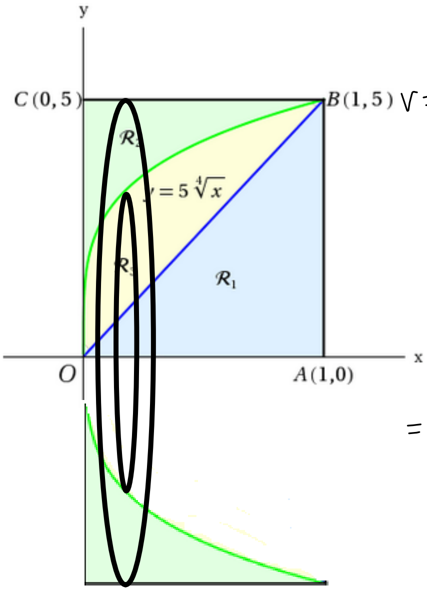
$$= \frac{25\pi}{2} \left[\sin(2x)\right]_0^{\frac{\pi}{4}} = \frac{25\pi}{2}\left[\sin\frac{\pi}{2}\right] = \boxed{\frac{25\pi}{2}}$$

Nov 20-10:12 AM

9. Question Details SCalc8 5.2.023. [3354024]

Refer to the figure and find the volume generated by rotating the given region about the specified line.

$\mathcal{R}_2$ about OA



$25(x^{\frac{1}{4}})^2 = 25x^{\frac{1}{2}}$

$V = \pi \int_0^1 \left(5^2 - (5\sqrt[4]{x})^2\right) dx$

$= \pi \int_0^1 (25 - 25x^{\frac{1}{2}}) dx$

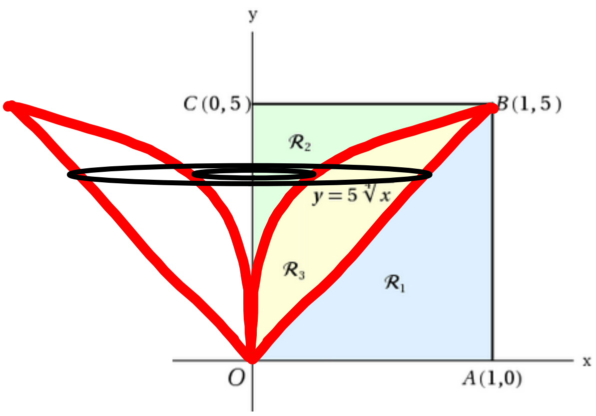
$= \pi \left[25x - \frac{50}{3} x^{\frac{3}{2}}\right]_0^1$

$= \pi \left[25 - \frac{50}{3}\right] = \boxed{\frac{25\pi}{3}}$

10. Question Details SCalc8 5.2.028

Refer to the figure and find the volume V generated by rotating the given region about the specified line.

$\mathcal{R}_3$ about OC



OB: $m = \frac{5}{1}$

$y = 5x \Rightarrow$

$x = 5y$

$y = 5\sqrt[4]{x}$

$x^{1/4} = \frac{y}{5}$

$x = \frac{y^4}{5^4}$

$\pi \int_0^5 \left((5y)^2 - \left(\frac{1}{5^4} y^4\right)^2\right) dy$

$= \pi \int_0^5 \left(25y^2 - \frac{1}{5^8} y^8\right) dy$

$= \pi \left[\frac{25}{3} y^3 - \frac{1}{9 \cdot 5^8} y^9\right]_0^5$

$= \pi \left[\frac{25 \cdot 5^3}{3} - \frac{1}{9 \cdot 5^8}(5^9)\right]$

$= \frac{3125\pi}{3} - \frac{5}{9} = \boxed{\frac{3125\pi}{3} - \frac{5}{9}}$

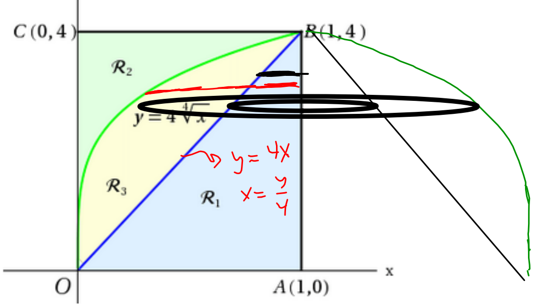
125
$\times 25$
625
2500
3125

11. Question Details SCalc8 5.2.029

Refer to the figure and find the volume V generated by rotating the given region about the specified line.

$\mathcal{R}_3$ about AB

Changed the picture slightly, for this one.



$y = 4\sqrt[4]{x}$

$\frac{y}{4} = \sqrt[4]{x}$

$\frac{y^4}{4^4} = x$

$\pi\int_0^4 \left(\left(\frac{y^4}{4^4}\right)^2 - \left(\frac{y}{4}\right)^2\right)dy = \pi\int_0^4 \left(\frac{y^8}{4^8} - \frac{y^2}{16}\right)dy = \pi\left[\frac{1}{9\cdot 4^8}y^9 - \frac{1}{3\cdot 16}y^3\right]_0^4$

$= \pi\left[\frac{1}{9\cdot 4^8}\cdot 4^9 - \frac{1}{3\cdot 4^2}\cdot 4^3\right]$

$= \pi\left[\frac{4}{9} - \frac{4}{3}\right] = \pi\left[\frac{12-4}{9}\right] \boxed{\frac{8\pi}{9} = V}$

12. Question Details SCalc8 5.2.033. [3353957]

Set up an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Then use your calculator to evaluate the integral correct to five decimal places.

$x^2 + 9y^2 = 9$ $\Rightarrow$ $\frac{x^2}{9} + y^2 = 1$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $a = 3,\ b = 1$

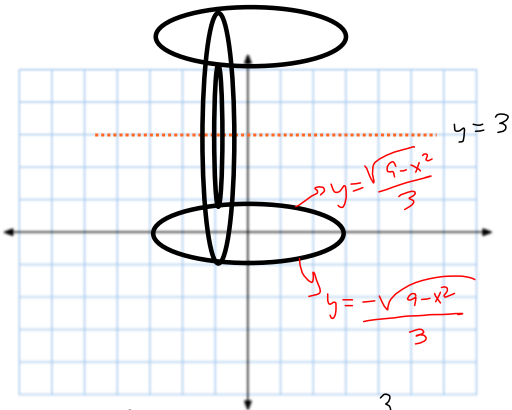
(a) About $y = 3$

(b) About $x = 3$

$y^2 = 1 - \frac{x^2}{9}$

$y = \pm\sqrt{1 - \frac{x^2}{9}}$

$= \frac{\pm\sqrt{9-x^2}}{3}$



$\pi\int_{-3}^{3}\left[\left(3 + \frac{\sqrt{9-x^2}}{3}\right)^2 - \left(3 - \frac{\sqrt{9-x^2}}{3}\right)^2\right]dx = 2\pi\int_0^3\left(\left(\frac{9+\sqrt{9-x^2}}{3}\right)^2 - \left(\frac{9-\sqrt{9-x^2}}{3}\right)^2\right)dx$

$= 2\pi\int_0^3\left(\frac{81 + 2\cdot 9\sqrt{9-x^2} + 9 - x^2}{9} - \frac{81 - 18\sqrt{9-x^2} + 9 - x^2}{9}\right)dx$

$= \frac{2\pi}{9}\int_0^3 36\sqrt{9-x^2}\,dx = 8\pi\int_0^3\sqrt{9-x^2}\,dx = 18\pi^2 \approx 177.6528793$

$\approx \boxed{177.65288}$

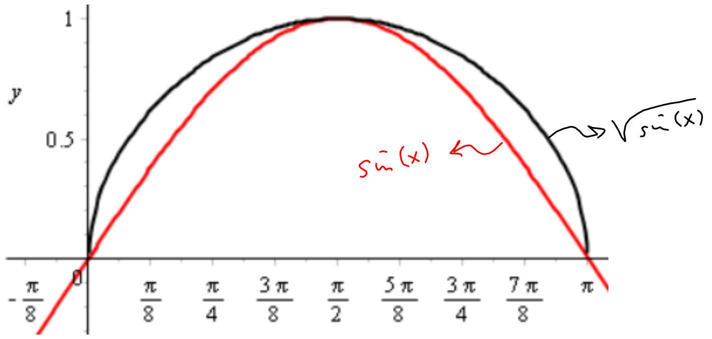
Nov 17-7:58 AM

13. Question Details SCalc8 5.2.039.

The integral represents the volume of a solid. Describe the solid.

$$\pi \int_0^{\pi} \sin(x)\, dx = \pi \int_0^{\pi} \left(\sqrt{\sin(x)}\right)^2 dx$$

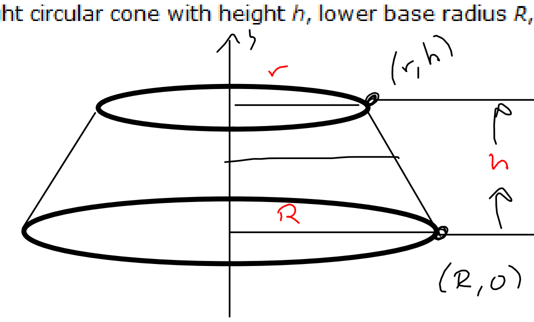
$y = \sqrt{\sin x}$ revolved about the x-axis.



14. Question Details SCalc8 5.2.048.

Find the volume V of the described solid S.

A frustum of a right circular cone with height h, lower base radius R, and top radius r



$$m = \frac{h-0}{r-R} = \frac{h}{r-R}$$

$$y = \frac{h}{r-R}(x-R)$$

$$= m(x-x_1)+y_1$$

$$= \frac{h}{r-R}x - \frac{h}{r-R}R$$

$$\frac{h}{r-R}x = y + \frac{h}{r-R}R$$

$$x = \frac{y(r-R)}{h} + R$$

= RADIUS for the rotation

$$\pi \int_0^h \left(\frac{y(r-R)}{h} + R \right) dy$$

$$= \pi \left[\left(\tfrac{1}{2}y^2\right)\left(\frac{r-R}{h}\right) + Ry \right]_0^h$$

$$= \pi \left[\frac{h^2}{2}\left(\frac{r-R}{h}\right) + Rh \right]$$

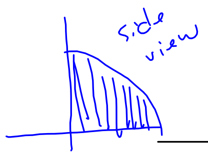
$$= \pi \left[\frac{h(r-R)}{2} + \frac{2Rh}{2} \right]$$

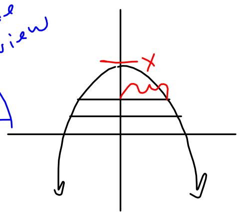
$$= \pi \left[\frac{hr - hR - 2Rh}{2} \right] = \boxed{\frac{\pi h}{2}\left[hr - 3Rh \right]}$$

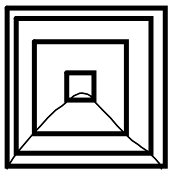
15. Question Details SCalc8 5.2.058.

Find the volume V of the described solid S.

The base of S is the region enclosed by the parabola $y = 2 - 3x^2$ and the x–axis. Cross-sections perpendicular to the y–axis are squares.







$y = 2 - 3x^2 \Rightarrow 3x^2 = 2 - y$

$x^2 = \frac{2-y}{3}$

$x = \pm\sqrt{\frac{2-y}{3}}$

$x = \sqrt{\frac{2-y}{3}}$

Area is x^2

$2x = 2\sqrt{\frac{2-y}{3}}$

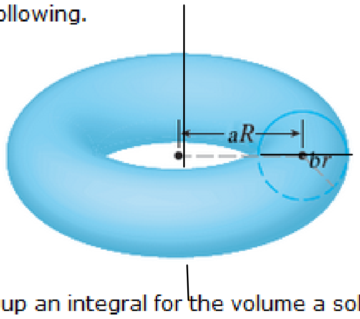
$= \left(2\sqrt{\frac{2-y}{3}}\right)^2 = 4\left(\frac{2-y}{3}\right) = \frac{1}{3}[8-4y]$

Vol = Area $\cdot \Delta x$

$V = \frac{1}{3}\int_0^3 (8-4y)\,dy = \frac{1}{3}\left[8y - 2y^2\right]_0^3 = \frac{1}{3}\left[24 - 18\right] = \boxed{2}$

16. Question Details SCalc8 5.2.063. [3353997]

Consider the following.



Eq'n of circle:

$(x-aR)^2 + (y^2) = (br)^2$

$y^2 = b^2r^2 - (x-aR)^2$

$y = \pm\sqrt{\sim\sim}$

we take the top half & Double it

(a) Set up an integral for the volume a solid *torus* (the donut-shaped solid shown in the figure) with radii *br* and *aR*. (Let *a* = 4 and *b* = 3.)

(b) By interpreting the integral as an area, find the volume *V* of the torus.

$y = \sqrt{b^2r^2 - (x-aR)^2}$

$(x-aR)^2 = b^2r^2 - y^2$

$x - aR = \pm\sqrt{b^2r^2-y^2}$

$x = aR \pm \sqrt{b^2r^2-y^2}$

$\pi\int \text{outer}^2 - \text{inner}^2$

$= 2\pi\int_0^{br}\left(\left(aR+\sqrt{b^2r^2-y^2}\right)^2 - \left(aR-\sqrt{b^2r^2-y^2}\right)^2\right)dy$

$= 2\pi\int_0^{br}\left(4aR\sqrt{b^2r^2-y^2}\right)dy$

$a=4$ & $b=3 \Rightarrow$

$$\boxed{2\pi\int_0^{3r} 16R\sqrt{9r^2-y^2}\,dy}$$