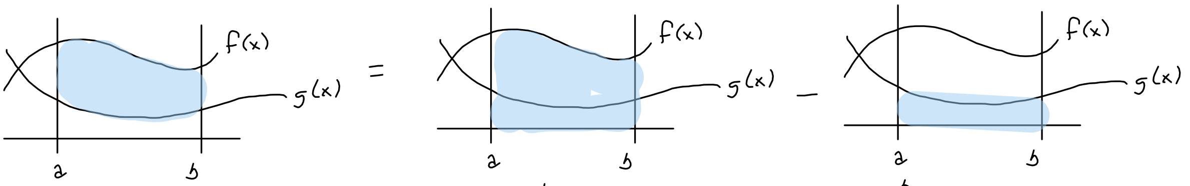
Section 5.1 - Area Between Curves

Some quick ideas and techniques



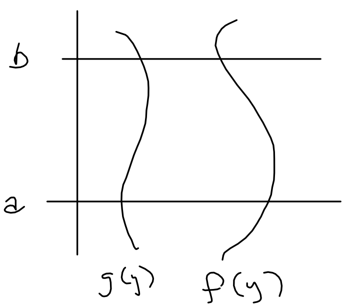
Area between f(x) & g(x) from x = a to x = b.

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Linearity of "$\int$"

$$= \int_a^b (f(x) - g(x))\,dx$$

Area between f(y) & g(y)



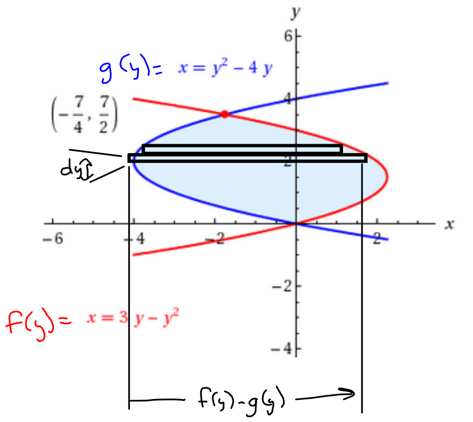
$$\int_a^b (f(y) - g(y))\,dy$$

For these, "right" means "above."
"Left" means "below"

When in doubt, do as f(x) & g(x), then turn it on its side!

1. Question Details SCalc8 5.1.004.

Find the area of the shaded region.



6 49
7
343

$$\int_0^{\frac{7}{2}} \left(f(y) - g(y)\right) dy$$

$$= \int_0^{\frac{7}{2}} \left((3y - y^2) - (y^2 - 4y)\right) dy$$

$$= \int_0^{\frac{7}{2}} (-2y^2 + 7y)\,dy = \left[-\frac{2}{3}y^3 + \frac{7}{2}y^2\right]_0^{\frac{7}{2}}$$

$$= -\frac{2}{3}\left(\frac{7}{2}\right)^3 + \frac{7}{2}\left(\frac{7}{2}\right)^2 - (0+0)$$

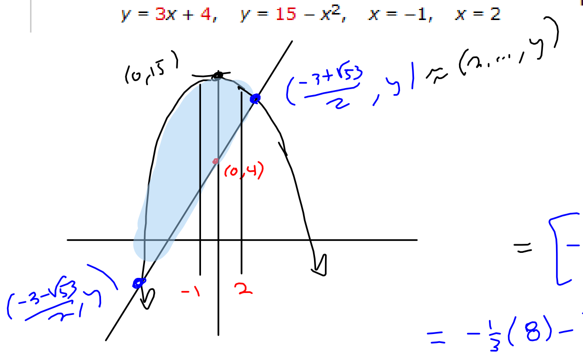
$$= \frac{1}{3}\left(\frac{343}{8}\right) = \boxed{\frac{343}{24}}$$

2. Question Details SCalc8 5.1.005. [335396

Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width.

$y = 3x + 4, \quad y = 15 - x^2, \quad x = -1, \quad x = 2$

Find the area of the region.



$$\int_{-1}^{2} (\text{upper} - \text{lower})$$

$15 - x^2 - (3x + 4)$

$= 15 - x^2 - 3x - 4$

$= -x^2 - 3x + 11$

$$= \int_{-1}^{2} (-x^2 - 3x + 11)\,dx$$

$$= \left[-\frac{1}{3}x^3 - \frac{3}{2}x^2 + 11x\right]_{-1}^{2} =$$

$$= -\frac{1}{3}(8) - \frac{3}{2}(4) + 11(2) - \left[-\frac{1}{3}(-1) - \frac{3}{2}(1) + 11(-1)\right]$$

$$= -\frac{8}{3} - 6 + 22 - \left[\frac{1}{3} - \frac{3}{2} - 11\right]$$

$$= -\frac{8}{3} + 16 - \frac{1}{3} + \frac{3}{2} + 11 \qquad = -3 + \frac{3}{2} + 27$$

$$= \frac{48 + 3}{2} = \boxed{\frac{51}{2}}$$

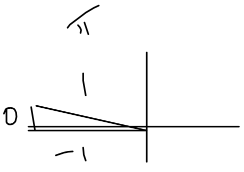
3. Question Details SCalc8 5.1.006. [3353664

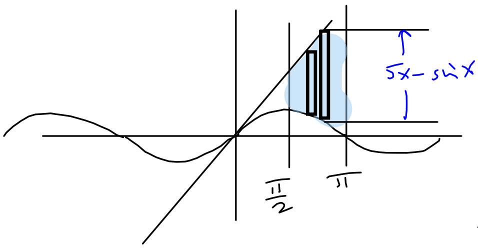
Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle.

$y = \sin(x)$, $y = 5x$, $x = \pi/2$, $x = \pi$

Find the area of the region.

Slope of $\sin x$ is $\cos(0) = 1 < 5 = m$ for $y = 5x$ @ $x = 0$





$$\int_{\frac{\pi}{2}}^{\pi} (5x - \sin x)\,dx$$

$$= \left[\frac{5}{2}x^2 + \cos x\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{5}{2}\cdot\pi^2 + \cos\pi - \left(\frac{5}{2}\cdot\frac{\pi^2}{4} + \cos\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{5\pi^2}{2} - 1 - \frac{5\pi^2}{8} - 0$$

$$= \boxed{\frac{15\pi^2}{8} - 1} = \frac{15\pi^2 - 8}{8}$$

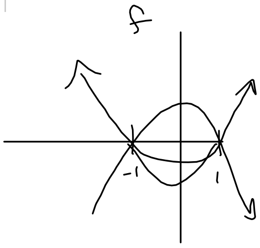
4. Question Details SCalc8 5.1.011.MI. [335390

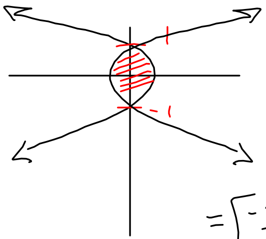
Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle.

$x = 7 - 7y^2$, $x = 7y^2 - 7$

f g

Find the area of the region.





$$\int_{-1}^{1} (f(y) - g(y))\,dy$$

$$= \int_{-1}^{1} (-14y^2 + 14)\,dy$$

$$= \left[\frac{-14y^3}{3} + 14y\right]_{-1}^{1}$$

$$= \frac{-14}{3} + 14 - \left(\frac{14}{3} - 14\right)$$

$$= \left(\frac{28}{3}\right)(2) = \boxed{\frac{56}{3}}$$

1 $y = -7x^2 + 7$

$= -7(x^2 - 1)$

$= -7(x-1)(x+1)$

$$\int_{-1}^{1} (f(x) - g(x))\,dx = \int_{-1}^{1} (-14x^2 + 14)\,dx$$

$y = 7x^2 - 7$

$= 7(x^2 - 1)$

$= 7(x-1)(x+1)$

5. Question Details SCalc8 5.1.013.

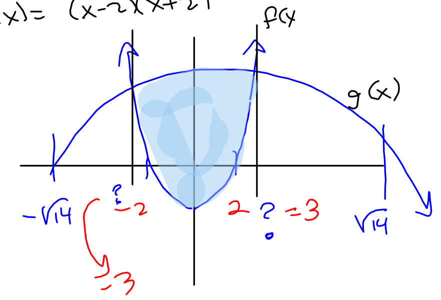
Sketch the region enclosed by the given curves.

$f(x) = y = 14 - x^2$, $y = x^2 - 4 = g(x)$

Find its area.

$f(x) = -(x^2 - 14) = -(x - \sqrt{14})(x + \sqrt{14})$

$g(x) = (x-2)(x+2)$



$f(x) = g(x)$

$14 - x^2 = x^2 - 4$

$18 - 2x^2 = 0$

$2x^2 - 18 = 0$

$2(x^2 - 9) = 0$

$2(x-3)(x+3) = 0$

$x = \pm 3$

$\text{Area} = \int g - f$

$= \int_{-3}^{3} \left(14 - x^2 - (x^2 - 4)\right) dx$

$= \int_{-2}^{2} (18 - 2x^2)\, dx$ Symmetry!

$= 2\int_0^2 (18 - 2x^2)\, dx = 2\left[18x - \frac{2}{3}x^3\right]_0^2$

$= 2\left[36 - \frac{16}{3}\right] = 2\left[\frac{108 - 16}{3}\right]$

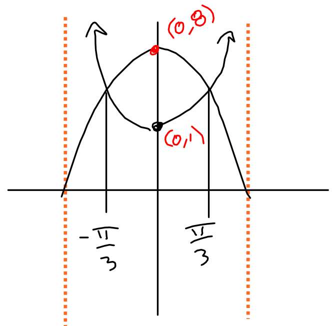
$= \frac{2(92)}{3} = \boxed{\frac{184}{3}}$

6. Question Details SCalc8 5.1.015. [339534

Sketch the region enclosed by the given curves.

$y = \sec^2(x)$, $y = 8\cos(x)$, $-\pi/3 \le x \le \pi/3$

Find its area.



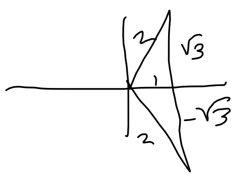
$8\cos x = \sec^2 x = \frac{1}{\cos^2 x}$

$8\cos^3 x = 1$

$\cos^3 x = \frac{1}{8}$

$\cos x = \frac{1}{2}$

$x = \arccos\left(\frac{1}{2}\right)$



$x = \frac{\pi}{3}, -\frac{\pi}{3}$

$\text{Area} = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (8\cos x - \sec^2 x)\, dx = 2\int_0^{\frac{\pi}{3}} (8\cos x - \sec^2 x)\, dx = 2\left[8\sin x - \tan x\right]_0^{\frac{\pi}{3}}$

$= 2\left[8\sin\frac{\pi}{3} - \tan\frac{\pi}{3} - (0 - 0)\right] = \frac{16\sqrt{3}}{2} - 2\sqrt{3} = 6\sqrt{3}$

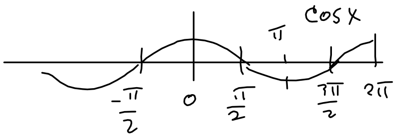
7. Question Details SCalc8 5.1.019.

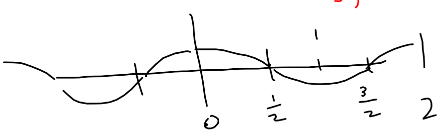
Sketch the region enclosed by the given curves.

$y = 7\cos(\pi x), \quad y = 8x^2 - 2$

Find its area.

$\cos x$ $\cos(\pi x)$ (Divide x's by π)

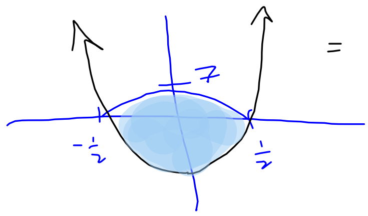




$8x^2 - 2 = 2(4x^2 - 1)$

$= 2(2x-1)(2x+1)$

$= 0 \quad @ \quad \pm\frac{1}{2}$



=

Area =

$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\cos(\pi x) - (8x^2 - 2)\right) dx$

$2\int_0^{\frac{1}{2}}$ ——— by Symm

$= 2\left(\frac{1}{\pi}\int_{0=x}^{\frac{1}{2}=x} \cos u\,du\right) - 2\int_0^{\frac{1}{2}} (8x^2 - 2)\,dx$

$= \frac{2}{\pi} \sin(\pi x)\Big]_0^{\frac{1}{2}} - 2\left[\frac{8}{3}x^3 - 2x\right]_0^{\frac{1}{2}}$

$\frac{8}{3}\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)$

$= \frac{8}{3} \cdot \frac{1}{8} - 1$

$= \frac{2}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] - 2\left[\frac{1}{3} - 1\right] \rightarrow -2\left[-\frac{2}{3}\right] = +\frac{4}{3}$

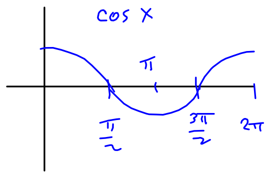
$= \boxed{\frac{2}{\pi} + \frac{4}{3}} = \frac{4\pi + 6}{3\pi} = \frac{2}{3}\left(\frac{\pi + 3}{\pi}\right)$

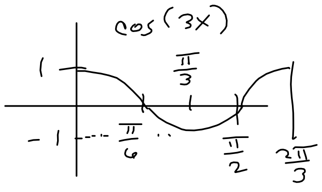
8. Question Details SCalc8 5.1.024.

Sketch the region enclosed by the given curves.

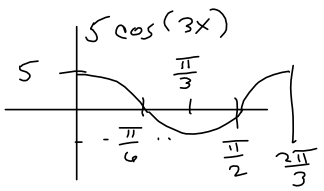
$y = 5\cos(3x), \quad y = 5 - 5\cos(3x), \quad 0 \le x \le \pi/3$

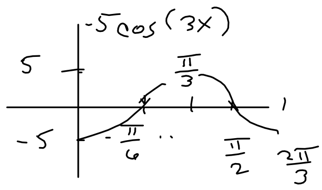
Find its area.

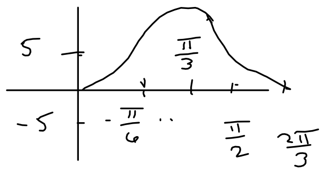


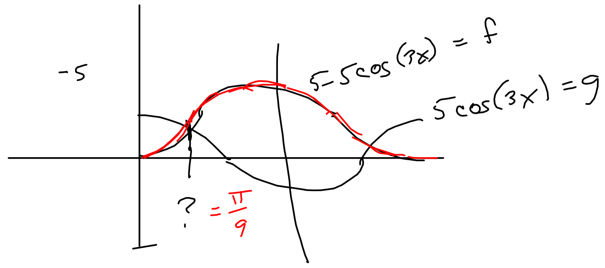


$-5\cos(3x)$









$5\cos(3x) = 5 - 5\cos(3x)$

$10\cos(3x) = 5$

$\cos(3x) = \frac{1}{2}$

$3x = \frac{\pi}{3}$

$x = \frac{\pi}{9}$

$$\text{Area} = \int_0^{\frac{\pi}{9}} g - f + \int_{\frac{\pi}{9}}^{\frac{\pi}{3}} f - g = \int_0^{\frac{\pi}{9}} \left(10\cos(3x) - 5\right)dx + \int_{\frac{\pi}{9}}^{\frac{\pi}{3}} \left(5 - 10\cos(3x)\right)dx$$

$$= \int_0^{\frac{\pi}{9}} 10\cos(3x)\,dx - \int_0^{\frac{\pi}{9}} 5\,dx + \int_{\frac{\pi}{9}}^{\frac{\pi}{3}} 5 - \int_{\frac{\pi}{9}}^{\frac{\pi}{3}} 10\cos(3x)\,dx$$

$u = 3x$

$du = 3dx$

$$= \frac{1}{3}\int_0^{\frac{\pi}{9}} 10\cos(3x)\,3dx - 5x\Big]_0^{\frac{\pi}{9}} + 5x\Big]_{\frac{\pi}{9}}^{\frac{\pi}{3}} - \frac{10}{3}\int_{\frac{\pi}{9}}^{\frac{\pi}{3}} \cos(3x)\,3dx$$

$$= \frac{10}{3}\left[\sin(3x)\right]_0^{\frac{\pi}{9}} - \frac{5\pi}{9} + \frac{5\pi}{3} - \frac{5\pi}{9} - \frac{10}{3}\left[\sin(3x)\right]_{\frac{\pi}{9}}^{\frac{\pi}{3}}$$

$$= \frac{10}{3}\left[\sin\left(\frac{\pi}{3}\right) - \sin(0)\right] - \frac{10\pi}{9} + \frac{5\pi}{3} - \frac{10}{3}\left[\sin(\pi) - \sin\left(\frac{\pi}{3}\right)\right]$$

$$= \frac{10}{3}\left[\frac{\sqrt{3}}{2}\right] + \frac{-10\pi + 15\pi}{9} - \frac{10}{3}\left[0 - \frac{\sqrt{3}}{2}\right] = \frac{5\sqrt{3}}{3} + \frac{5\pi}{9} + \frac{5\sqrt{3}}{3} = \boxed{\frac{5\pi}{9} + \frac{10\sqrt{3}}{3}}$$

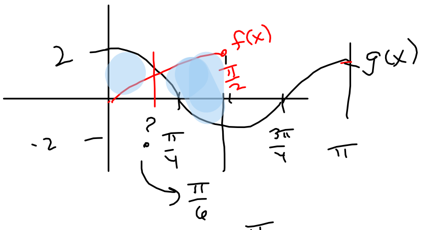
$$= \frac{5\pi + 30\sqrt{3}}{9}$$

9. Question Details SCalc8 5.1.035.

Evaluate the integral and interpret it as the area of a region.

$$\int_0^{\pi/2} |2\sin(x) - 2\cos(2x)|\,dx = \int_a^b |f-g|$$

Sketch the region.



$f - g$

$2\sin x - 2\cos(2x) = 0$

$\sin x - \cos(2x) = 0$

$\sin(x) - (1 - 2\sin^2(x)) = 0$

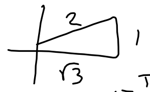
$+2\sin^2 x + \sin x - 1$

$2u^2 + u - 1$

$(2u-1)(u+1)$

$2\sin x = 1$ $\quad\sin x = -1$ Don't care.

$\sin x = \frac{1}{2}$



$x = \frac{\pi}{6}$

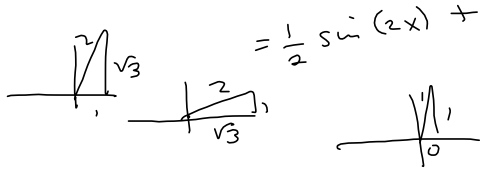
$$\int_0^{\frac{\pi}{6}} g - f + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \int_0^{\frac{\pi}{6}} (2\cos(2x) - 2\sin(x))\,dx$$

$$+ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin x - 2\cos(2x))\,dx$$

$$\int \cos(2x)\,dx = \frac{1}{2}\int \cos(2x)\,2\,dx$$

$$= \frac{1}{2}\sin(2x) + C$$



$$= \sin(2x)\Big]_0^{\frac{\pi}{6}} + 2\cos(x)\Big]_0^{\frac{\pi}{6}}$$

$$= -\sin(2x)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - 2\cos(x)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \sin\left(\frac{\pi}{3}\right) - \sin(0) + 2\cos\left(\frac{\pi}{6}\right) - 2\cos(0) - \sin\pi + \sin\frac{\pi}{3} - 2\cos\frac{\pi}{2} + 2\cos\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} - 0 + \frac{2\sqrt{3}}{2} - 2 - 0 + \frac{\sqrt{3}}{2} - 0 + \frac{2\sqrt{3}}{2}$$

$$= \sqrt{3} + \sqrt{3} + \sqrt{3} - 2 = 3\sqrt{3} - 2$$

Nov 15-10:07 AM

Nov 16-2:18 PM