§4.5 Substitution Rule

Recall Chain Rule

$$\frac{d}{dx}\left[\sin(x^2-5x)\right]$$

$f(u) = \sin(u)$

$u(x) = x^2 - 5x$

$$\frac{d}{dx}\left[f(u(x))\right] = \frac{df}{du}\cdot\frac{du}{dx}$$

$$= \cos(x^2-5x)\cdot(2x-5)$$

$$\int \cos(x^2-5x)\cdot(2x-5)\,dx = \sin(x^2-5x) + C$$

$$\frac{d}{dx}\left[\sqrt{\sin(x^2-5x)}\right]$$

$$\frac{d}{dx}\left[f(u(g(x)))\right] = \frac{df}{du}\cdot\frac{du}{dg}\cdot\frac{dg}{dx}$$

$f(u) = \sqrt{u} = u^{\frac{1}{2}} \rightsquigarrow \frac{1}{2}u^{-\frac{1}{2}}$

$u(g) = \sin(g) \rightsquigarrow \cos(g)$

$g(x) = x^2-5x \rightsquigarrow 2x-5$

$$= \frac{1}{2}\left(\sin(x^2-5x)\right)^{-\frac{1}{2}}\cdot\cos(x^2-5x)\cdot(2x-5)$$

$$\int \frac{1}{2}(2x-5)\cos(x^2-5x)\sin(x^2-5x)\,dx$$

$$= \sqrt{\sin(x^2-5x)} + C$$

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

Look for "u" inside & then "du"

6. $\displaystyle\int \frac{\sec^2(1/x)}{x^2}\,dx$

$u = \frac{1}{x} = x^{-1}$

$\Rightarrow du = -x^{-2}\,dx$

$= -\frac{1}{x^2}\,dx$

$dx = -x^2\,du$

$\displaystyle\int \sec^2\left(\tfrac{1}{x}\right)\cdot\frac{1}{x^2}\,dx = \int \sec^2(u)\cdot\frac{1}{x^2}\cdot(-x^2\,du)$

$\displaystyle = -\int \sec^2(u)\,du = -\tan(u) + C$

$= -\tan\left(\tfrac{1}{x}\right) + C$

16. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin(u)\cdot\frac{1}{\sqrt{x}}\cdot 2\sqrt{x}\,du$

$u = x^{\frac{1}{2}} \Rightarrow du = \frac{1}{2}x^{-\frac{1}{2}}dx$

$du = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}\,dx$

$2\sqrt{x}\,du = dx$

$\displaystyle = 2\int \sin(u)\,du = -2\cos(u) + C = -2\cos(\sqrt{x}) + C$

22. $\displaystyle\int \frac{\cos(\pi/x)}{x^2}\,dx$

10. $\displaystyle\int (3t+2)^{2.4}\,dt$

$u = \frac{\pi}{x} = \pi x^{-1} \Rightarrow du = -\pi x^{-2}dx \Rightarrow -\frac{1}{\pi}x^2\,du = dx$

$\displaystyle = \int \cos\left(\frac{\pi}{x}\right)\cdot\frac{1}{x^2}\,dx = \int \cos\left(\frac{\pi}{x}\right)x^{-2}\,dx$

$\displaystyle = \int \cos(u)\,x^{-2}\cdot -\frac{1}{\pi}x^2\,du$

$\displaystyle = -\frac{1}{\pi}\int \cos(u)\,du$ etc.

$\boxed{5}$ **The Substitution Rule for Definite Integrals** If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

* #s 37, 51 incorrect on homework

37. * $\int_0^1 \sqrt[3]{1 + 7x}\,dx$

$u = 7x+1$

$\Rightarrow du = 7dx$

$\Rightarrow dx = \frac{1}{7}du$

$$= \int_{x=0}^{x=1} (7x+1)^{\frac{1}{3}}\,dx = \int_{x=0}^{x=1} u^{\frac{1}{3}} \cdot \frac{1}{7}du = \frac{1}{7}\int_{x=0}^{x=1} u^{\frac{1}{3}}du = \frac{1}{7}\left[\frac{u^{\frac{4}{3}}}{\frac{4}{3}}\right]_{x=0}^{x=1}$$

$$= \frac{1}{7}\left[\frac{3}{4}(7x+1)^{\frac{4}{3}}\right]_{x=0}^{x=1}$$

$$= \frac{3}{28}\left[8^{\frac{4}{3}} - 1^{\frac{4}{3}}\right] = \frac{3}{28}\cdot\left[16-1\right]$$

$$= \boxed{\frac{45}{28}}$$

$8^{\frac{4}{3}} = \left(8^{\frac{1}{3}}\right)^4 = 2^4 = 16$

* Optional: Replace x in limits of integration by u.

$u = 7x+1$

$u(1) = 7+1 = 8$

$u(0) = 1$

So $\frac{1}{7}\int_{x=0}^{x=1} u^{1/3}du = \frac{1}{7}\int_{u=0}^{u=1} u^{\frac{1}{3}}du = \frac{1}{7}\int_0^1 u^{\frac{1}{3}}du$

47. $\int_1^2 x\sqrt{x-1}\,dx$

$u = x-1 \Rightarrow du = 1\,dx$

$$\int_{x=1}^{x=2} x\sqrt{u}\,du = \int_{x=1}^{x=2} (u+1)u^{\frac{1}{2}}\,du$$

$x = u+1$

$$= \int_{x=1}^{x=2} \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du = \int_{0=u}^{1=u} \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$

$u(1) = 1-1 = 0$

$u(2) = 2-1 = 1$

$$= \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]_0^1 = \frac{2}{5} + \frac{2}{3} - (0+0) = \frac{6+10}{15} = \frac{16}{15}$$

51. $\displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

$u = x$

$u = \sqrt{x}$

$u = \sqrt{x} + 1$
$= x^{\frac{1}{2}} + 1$

$\Rightarrow du = \frac{1}{2} x^{-\frac{1}{2}}\, dx$

$dx = 2x^{\frac{1}{2}} du$

$u = x^{\frac{1}{2}} + 1$, so

$x^{\frac{1}{2}} = u - 1$

$2x^{\frac{1}{2}} = 2u - 2 = 2(u-1)$

$u(0) = 1$

$u(1) = 2$

$$\int_{x=0}^{x=1} (\sqrt{x}+1)^{-4}\, dx$$

$$= \int_{x=0}^{x=1} u^{-4} \left(2x^{\frac{1}{2}} du\right)$$

$$= 2\int_{x=0}^{x=1} u^{-4}(u-1)\,du = 2\int_1^2 \left(u^{-3} - u^{-4}\right) du = 2\left[\frac{u^{-2}}{-2} - \frac{u^{-3}}{-3}\right]_1^2$$

$$= 2\left[\frac{\frac{1}{4}}{-2} - \frac{\frac{1}{8}}{-3} - \left(\frac{1}{-2} - \frac{1}{-3}\right)\right]$$

$$= 2\left[-\frac{1}{8} + \frac{1}{24} + \frac{1}{2} - \frac{1}{3}\right]$$

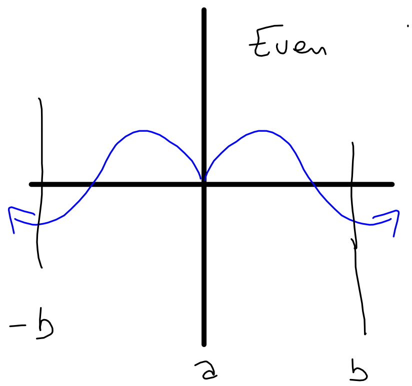
$$= -\frac{1}{4} + \frac{1}{12} + 1 - \frac{2}{3}$$

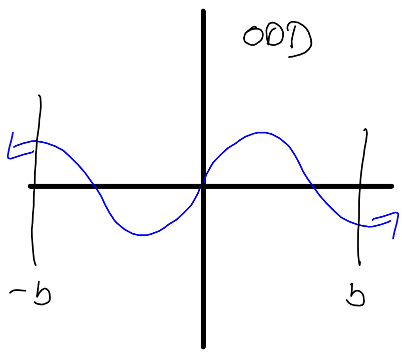
$$= -\frac{3}{12} + \frac{1}{12} + \frac{12}{12} - \frac{8}{12} = \frac{-3+1+12-8}{12} = \frac{2}{12} = \boxed{\frac{1}{6}}$$

$\boxed{6}$ **Integrals of Symmetric Functions** Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.





x^{2n} even $+$ sin x odd $-$

x^{2n+1} odd $-$ cos x even $+$

c even $+$

$x^6 \sin x$

$(+)(-) \longrightarrow -$

$$\frac{x^2 \tan x}{x^3 - 5x^3 + \sin x} : \frac{(+)\frac{\sin x}{\cos x}}{-} = \frac{+ \cdot \frac{-}{+}}{-} = \frac{-}{-}$$

$= +$