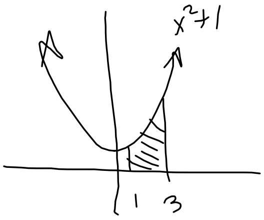
Find area under x^2+1 between $x=1$ and $x=3$



We find exact area as limit of a Riemann sum.

$[a,b]=[1,3] \Longrightarrow \Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n} = \Delta x$

$x_k = a + k\Delta x = 1 + k\cdot\frac{2}{n} = 1 + \frac{2k}{n}$

$$* \quad R_n = \sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n}(x_k^2+1)\Delta x = \Delta x \sum_{k=1}^{n}\left(\left(\frac{2k}{n}+1\right)^2+1\right) = \frac{2}{n}\sum_{k=1}^{n}\left(\left(\frac{2k}{n}+1\right)^2+1\right)$$

$$* \quad = \frac{2}{n}\sum_{k=1}^{n}\left(\frac{4k^2}{n^2}+\frac{4k}{n}+1+1\right) = \frac{2}{n}\sum_{k=1}^{n}\left(\frac{4k^2}{n^2}+\frac{4k}{n}+2\right)$$

$$* \quad \frac{2}{n}\left[\sum_{k=1}^{n}\frac{4k^2}{n^2}+\sum_{k=1}^{n}\frac{4k}{n}+\sum_{k=1}^{n}2\right] = \frac{2}{n}\left[\frac{4}{n^2}\sum_{k=1}^{n}k^2+\frac{4}{n}\sum_{k=1}^{n}k+\sum_{k=1}^{n}2\right]$$

$$* \quad = \frac{8}{n^3}\sum_{k=1}^{n}k^2 + \frac{8}{n^2}\sum_{k=1}^{n}k + \frac{2}{n}\sum_{k=1}^{n}2 = \frac{8}{n^3}\left(\frac{n^3+\dots}{3}\right)+\frac{8}{n^2}\left(\frac{n^2+\dots}{2}\right)+\frac{2}{n}\cdot 2n$$

$$\xrightarrow{n\to\infty} \frac{8}{3}+4+4$$

$$= \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{2}{n}\cdot 2n$$

Scratch

$$\left(\frac{2k}{n}+1\right)^2 = \left(\frac{2k}{n}\right)^2 + 2\left(\frac{2k}{n}\right)(1) + 1^2 = \frac{4k^2}{n^2}+\frac{4k}{n}+1$$

$$(a+b)^2 = a^2+2ab+b^2$$

$$= \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{2}{n}\cdot 2n$$

$$= \left(\frac{8}{n^3}\right)\left(\frac{2n^3+3n^2+n}{6}\right) + \left(\frac{8}{n^2}\right)\left(\frac{n^2+n}{2}\right) + 4$$

$$= \frac{16n^3}{6n^3} + \frac{24n^2}{6n^3} + \frac{8n}{6n^3} + \frac{8n^2}{2n^2} + \frac{8n}{2n^2} + 4$$

$$= \frac{16}{6} + \frac{4}{n} + \frac{4}{3n^2} + 4 + \frac{4}{n} + 4 \xrightarrow{n\to\infty} \frac{8}{3} + 4 + 4 = \frac{8+24}{3} = \boxed{\frac{32}{3}}$$

$\frac{4}{n} \to 0$, $\frac{4}{3n^2} \to 0$, $\frac{4}{n} \to 0$

$$n(n+1)(2n+1) = (n^2+n)(2n+1) = 2n^3 + n^2 + 2n^2 + n$$
$$= 2n^3 + 3n^2 + n$$

$$= \frac{8}{n^3}\sum_{k=1}^{n} k^2 + \frac{8}{n^2}\sum_{k=1}^{n} k + \frac{2}{n}\sum_{k=1}^{n} 2$$

$$= \frac{8}{n^3}\cdot\frac{2n^3 + \text{smaller}}{6} + \frac{8}{n^2}\cdot\frac{n^2 + \sim}{2} + \frac{2}{n}\cdot 2n$$

$$* \quad = \frac{8}{n^3}\cdot\frac{n^3+\sim}{3} + \frac{8}{n^2}\cdot\frac{n^2+\sim}{2} + 4 \xrightarrow{n\to\infty} \frac{8}{3} + \frac{8}{2} + 4$$

$$= \frac{8}{n^3}\cdot\frac{n^3}{3} + \frac{\sim}{3n^3} + \frac{8}{n^2}\cdot\frac{n^2}{2} + \frac{8\sim}{2n^2} + 4$$

$$\xrightarrow{n\to\infty} \frac{8}{3} + 4 + 4$$