Section 4.5 The Substitution Rule (Chain Rule in Reverse!)

u-substitution

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x)) \cdot g'(x) = \frac{dF}{dg} \cdot \frac{dg}{dx}$$

$$\frac{d}{dx}\left[\sqrt{x^2+1}\right] = \frac{d}{dx}\left[(x^2+1)^{\frac{1}{2}}\right] = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+1}}$$

$$\int \frac{x}{\sqrt{x^2+1}}\,dx \qquad g = x^2+1 \qquad \frac{dg}{dx} = 2x$$

$$\int \frac{x}{\sqrt{g(x)}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2+1}}\,dx$$

$$F(g(x)) = \frac{1}{\sqrt{x^2+1}} \qquad g'(x) = 2x$$

$$g(x) = x^2+1$$

$$\frac{dg}{dx} = 2x$$

$$dg = 2x\,dx \longrightarrow \text{STUDENTS}$$

$$\frac{dg}{2x} = dx$$

$$\int \frac{x}{\sqrt{g(x)}}\frac{dg}{2x} = \int g(x)^{-\frac{1}{2}}\frac{dg}{2} = \frac{1}{2}\int g(x)^{-\frac{1}{2}}dg$$

$$= \frac{1}{2}\int u^{-\frac{1}{2}}du = \frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= u^{\frac{1}{2}} + C$$

$$= \sqrt{x^2+1} + C$$

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du = F(u) + C$$

$$\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C$$

5 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

→ is cool, but often quicker to un-substitute before evaluating upper & lower.

Nov 9-10:16 AM

1. Question Details SCalc8 4.5.001.

Evaluate the integral by making the given substitution. (Use C for the constant of integration.)

$\int \cos(7x)\,dx, \quad u = 7x$ $\quad \frac{du}{dx} = 7 \Rightarrow du = 7dx$

$= \frac{1}{7}\int \cos(7x)\,7dx = \frac{1}{7}\int \cos(u)\,du = \frac{1}{7}\sin(u) + C = \boxed{\frac{1}{7}\sin(7x) + C}$

STUDENTS SEEM TO LIKE

$du = 7dx \Rightarrow dx = \frac{du}{7} \Rightarrow$

$\int \cos(u)\frac{du}{7} = \frac{1}{7}\int \cos(u)\,du$

2. Question Details SCalc8 4.5.003.MI.

Evaluate the integral by making the given substitution. (Use C for the constant of integration.)

$\int x^2\sqrt{x^3 + 35}\,dx, \quad u = x^3 + 35$

$du = 3x^2 dx$

Notice the "u" is "what's inside," just like chain rule?

M1

$= \frac{1}{3}\int (x^3+35)^{\frac{1}{2}}(3x^2 dx)$

$= \frac{1}{3}\int u^{\frac{1}{2}}du = \frac{1}{3}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(x^3+35)^{\frac{3}{2}} + C$

$= \boxed{\frac{2}{9}(x^3+35)^{\frac{3}{2}} + C}$ is fine

M2:

$3x^2dx = du$

$dx = \frac{du}{3x^2}$

$\int x^2\sqrt{u}\,\frac{du}{3x^2} = \frac{1}{3}\int u^{\frac{1}{2}}du$

$\left(\sqrt{x^3+35}\right)^3 = \sqrt{(x^3+35)^3}$

$= (x^3+35)\sqrt{x^3+35}$, etc

3. Question Details SCalc8 4.5.004.

Evaluate the integral by making the given substitution. (Use C for the constant of integration.)

$\int \sin^4(\theta)\cos(\theta)\,d\theta, \quad u = \sin(\theta) \Rightarrow du = \cos(\theta)\,d\theta$

$\frac{du}{\cos\theta} = d\theta$

$\int u^4 \cos\theta \frac{du}{\cos\theta}$

$= \int u^4 du = \frac{u^5}{5} + C$

$= \frac{1}{5}\sin^5(\theta) + C$

4. Question Details SCalc8 4.5.007

Evaluate the indefinite integral. (Use C for the constant of integration.)

$\int x\sqrt{8 - x^2}\,dx$ $\quad u = 8 - x^2 \Rightarrow du = -2x\,dx$

$= -\frac{1}{2}\int \left(\sqrt{8-x^2}\right)(-2x\,dx) = -\frac{1}{2}\int u^{\frac{1}{2}}du = -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = -\frac{1}{3}(8-x^2)^{\frac{3}{2}} + C$

5. Question Details SCalc8 4.5.009.

Evaluate the indefinite integral. (Use C for the constant of integration.)

$$\int (7-2x)^7\,dx \qquad \left(u = 7-2x \Rightarrow du = -2\,dx \right) \qquad dx = \frac{du}{-2}$$

$$= -\frac{1}{2}\int (7-2x)^7(-2\,dx) = -\frac{1}{2}\int u^7\,du = -\frac{1}{2}\frac{u^8}{8}$$

$$\int u^7 \frac{du}{-2} = -\frac{1}{2}\int u^7\,du$$

$$= -\frac{1}{2}\frac{(7-2x)^8}{8} + C$$

$$= \boxed{-\frac{1}{16}(7-2x)^8 + C}$$

6. Question Details SCalc8 4.5.010.

Evaluate the indefinite integral. (Use C for the constant of integration.)

$$\int \sin(t)\sqrt{1+\cos(t)}\,dt \qquad u = 1+\cos(t) \Rightarrow du = -\sin(t)\,dt$$

$$= -\int \left(\sqrt{1+\cos(t)}\right)(-\sin(t)\,dt) = -\int (u)^{\frac{1}{2}}(du) = -\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = -\frac{2}{3}u^{\frac{3}{2}} + C$$

$$-\frac{2}{3}(1+\cos(t))^{\frac{3}{2}} + C$$

7. Question Details SCalc8 4.5.017.

Evaluate the indefinite integral. (Use C for the constant of integration.)

$$\int \sec^2(\theta)\tan^4(\theta)\,d\theta \qquad u = \tan\theta \Rightarrow du = \sec^2\theta\,d\theta$$

$$= \int (\tan^4\theta)(\sec^2\theta\,d\theta) = \int u^4\,du = \frac{u^5}{5} + C = \boxed{\frac{1}{5}\tan^5\theta + C}$$

8. Question Details SCalc8 4.5.022.

Evaluate the indefinite integral. (Use C for the constant of integration.)

$$\int \frac{\cos(\pi/x^{33})}{x^{34}}\,dx \qquad u = \frac{\pi}{x^{33}} = \pi x^{-33} \Rightarrow du = -33\pi x^{-34}\,dx = \frac{-33\pi}{x^{34}}\,dx$$

$$-\frac{1}{33\pi}\int \cos(\pi x^{-33})(-33\pi x^{-34}\,dx) = -\frac{1}{33\pi}\int \cos(u)\,du = -\frac{1}{33\pi}\sin(u) + C$$

$$du = -33\pi x^{-34}\,dx$$

$$dx = \frac{x^{34}\,du}{-33\pi} \quad \text{OR} \quad \frac{du}{-33\pi x^{-34}}$$

$$= -\frac{1}{33\pi}\sin\left(\frac{\pi}{x^{33}}\right) + C$$

$$\int \frac{\cos\left(\frac{\pi}{x^{33}}\right)}{x^{34}} \cdot \frac{x^{34}\,du}{-33\pi} = -\frac{1}{33\pi}\int \cos\left(\frac{\pi}{x^{33}}\right)du$$

9. Question Details SCalc8 4.5.029.

Evaluate the indefinite integral. (Use C for the constant of integration.)

$$\int x(6x+7)^8\,dx \qquad u = 6x+7 \Rightarrow du = 6dx$$

what about the extra 'x'?

$u = 6x+7 = u$

$6x = u-7$

$x = \frac{u-7}{6}$

$$= \frac{1}{6}\int x\,(u^8)(6dx)$$

$$= \frac{1}{6}\int\left(\frac{u-7}{6}\right)u^8\,du$$

$$= \frac{1}{36}\int(u-7)u^8\,du = \frac{1}{36}\int(u^9 - 7u^8)\,du = \frac{1}{36}\left[\frac{(6x+7)^{10}}{10} - \frac{7}{9}(6x+7)^9\right] + C$$

$$= \frac{1}{36}\left[\frac{u^{10}}{10} - \frac{u^9}{9}\right] + C$$

10. Question Details SCalc8 4.5.030.

Evaluate the indefinite integral. (Use C for the constant of integration.)

$$\int x^3\sqrt{x^2+7}\,dx \qquad u = x^2+7 \Rightarrow du = 2x\,dx \rightarrow \text{So, } u-7 = x^2$$

$$= \frac{1}{2}\int x^2\left(\sqrt{x^2+7}\right)(2x\,dx) = \frac{1}{2}\int x^2 u^{\frac{1}{2}}\,du = \frac{1}{2}\int(u-7)u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}(u-7)\,du = \frac{1}{2}\int\left(u^{\frac{3}{2}} - 7u^{\frac{1}{2}}\right)du = \frac{1}{2}\left[\frac{2}{5}u^{\frac{5}{2}} - 7\left(\frac{2}{3}\right)u^{\frac{3}{2}}\right] + C$$

$$= \boxed{\frac{1}{5}(x^2+7)^{5/2} - \frac{14}{3}(x^2+7)^{\frac{3}{2}} + C}$$

11. Question Details SCalc8 4.5.042.

Evaluate the definite integral.

$$\int_0^{\pi/8}\cos(4x)\sin(\sin(4x))\,dx \qquad u = \sin(4x) \Rightarrow du = 4\cos(4x)\,dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{8}}\sin(\sin(4x))\,(4\cos(4x)\,dx) = \frac{1}{4}\int_{x=0}^{x=\frac{\pi}{8}}\sin(u)\,du \overset{\text{Book}}{=} \frac{1}{4}\int_{u(0)}^{u(\frac{\pi}{8})}\sin(u)\,du$$

$u(\frac{\pi}{8}) = \sin(4\cdot\frac{\pi}{8}) = \sin\frac{\pi}{2} = 1$

$u(0) = \sin(4\cdot 0) = \sin(0) = 0$

$$= \frac{1}{4}\int_0^1\sin(u)\,du = \frac{1}{4}(-\cos(u))\Big]_0^1$$

$$= -\frac{1}{4}\cos(1) + \frac{1}{4}\cos(0) = \boxed{-\frac{1}{4}\cos(1) + \frac{1}{4}}$$

5 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

I used "u" instead of "g"

Students seem to like this 2nd way, better.

$$= \frac{1}{4}\int_0^{\frac{\pi}{8}}\sin(\sin(4x))\,(4\cos(4x)\,dx) = \frac{1}{4}\int_{x=0}^{x=\frac{\pi}{8}}\sin(u)\,du = \frac{1}{4}(-\cos(u))\Big]_{x=0}^{x=\frac{\pi}{8}}$$

$$= -\frac{1}{4}\cos(\sin(4x))\Big]_{x=0}^{x=\frac{\pi}{8}} = -\frac{1}{4}\left[\cos\left(\sin\left(4\cdot\frac{\pi}{8}\right)\right) - \cos(\sin(4\cdot 0))\right]$$

$$= -\frac{1}{4}\left[\cos\left(\sin\left(\frac{\pi}{2}\right)\right) - \cos(\sin(0))\right] = -\frac{1}{4}\cos(1) + \frac{1}{4}\cos(0)$$

$$= \boxed{-\frac{1}{4}\cos(1) + \frac{1}{4}}$$

12. Question Details SCalc8 4.5.044

Evaluate the definite integral. (Assume $a > 0$.) (a is constant.)

$$\int_0^{a^{2/9}} x^8\sqrt{a^2 - x^9}\,dx$$

$u = a^2 - x^9 \Rightarrow du = -9x^8$

$$= -\frac{1}{9}\int_0^{a^{2/9}} (a^2 - x^9)^{\frac{1}{2}}(-9x^8dx) = -\frac{1}{9}\int_{x=0}^{x=a^{2/9}} u^{\frac{1}{2}}du = -\frac{1}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{x=0}^{x=a^{2/9}}$$

$u = 0$, $u = a^2$

$$= -\frac{2}{27}\left[(a^2 - x^9)^{\frac{3}{2}}\right]_0^{a^{2/9}} = -\frac{2}{27}\left[\left(a^2 - \left(a^{\frac{2}{9}}\right)^9\right)^{\frac{3}{2}} - (a^2 - 0^9)^{\frac{3}{2}}\right]$$

$$= -\frac{2}{27}\left[(a^2 - a^2)^{\frac{3}{2}} - (a^2)^{\frac{3}{2}}\right] = -\frac{2}{27}\left[0 - a^3\right] = \boxed{\frac{2a^3}{27}}$$

13. Question Details SCalc8 4.5.048.

Evaluate the definite integral.

$$\int_0^{10} \frac{x}{\sqrt{9 + 4x}}\,dx$$

$u = 9 + 4x \Rightarrow du = 4dx$

Need to get rid of x.

$$= \frac{1}{4}\int_0^{10} (9+4x)^{-\frac{1}{2}}(x)(4dx) = \frac{1}{4}\int_{x=0}^{x=10} u^{-\frac{1}{2}}x\,du$$

$u = 4x + 9 = u$

$4x = u - 9$

$x = \frac{u-9}{4}$

$$= \frac{1}{4}\int_{x=0}^{x=10} u^{-\frac{1}{2}}\left(\frac{u-9}{4}\right)du = \frac{1}{16}\int_{x=0}^{x=10}\left(u^{\frac{1}{2}} - 9u^{-\frac{1}{2}}\right)du$$

$u^{-\frac{1}{2}}(u-9) = u^{\frac{1}{2}} - 9u^{-\frac{1}{2}}$

$$= \frac{1}{16}\left[\frac{2}{3}u^{\frac{3}{2}} - 18u^{-\frac{1}{2}}\right]_{0=x}^{x=10} = \frac{1}{16}\left[\frac{2}{3}(4x+9)^{\frac{3}{2}} - 18(4x+9)^{-\frac{1}{2}}\right]_0^{10}$$

$$= \frac{1}{16}\left[\frac{2}{3}(49)^{\frac{3}{2}} - 18\left(49^{-\frac{1}{2}}\right) - \left[\frac{2}{3}(9)^{\frac{3}{2}} - 18\left(9^{-\frac{1}{2}}\right)\right]\right]$$

$49 \times 7 = 343$

$$= \frac{1}{16}\left[\frac{2}{3}(7)^3 - 18\left(\frac{1}{7}\right) - \left[\frac{2}{3}(27) - 18\left(\frac{1}{3}\right)\right]\right]$$

$$= \frac{1}{16}\left[\frac{2}{3}(343) - \frac{18}{7} - [18 - 6]\right] = \left[\frac{686}{3} - \frac{18}{7} - 12\right]\left(\frac{1}{16}\right)$$

$=$ I messed up, somewhere on the arithmetic. Show me where, for free points

$$\int_0^{10} \frac{x}{\operatorname{sqrt}(4\cdot x + 9)}\,dx = \frac{26}{3}$$

Check!

Nov 9-10:49 AM

14. Question Details SCalc8 4.5.055. [335397...

Evaluate $\int_{-4}^{4} (x+2)\sqrt{16-x^2}\,dx$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

$y = \sqrt{16-x^2}$

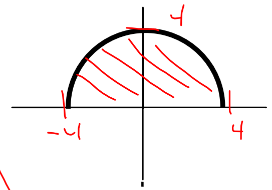
$y^2 = 16-x^2$

$x^2+y^2 = 16$

Times 2

$\int_{-4}^{4} x\sqrt{16-x^2}\,dx + 2\int_{-4}^{4}\sqrt{16-x^2}\,dx$

Gives area of a circle of radius $r=4$



$= \int_{-4}^{4} \text{ODD FUNC} + \pi(4)^2 = \boxed{16\pi}$

15. Question Details SCalc8 4.5.059.

If f is continuous and $\int_0^{10} f(x)\,dx = 8$, find $\int_0^5 f(2x)\,dx$.

$u = 2x \Rightarrow$

$du = 2dx$

$u(0) = 2(0) = 0$

$u(5) = 2(5) = 10$

5 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ $(u = g(x))$

This gives $\frac{1}{2}\int_0^5 f(2x)(2dx) = \frac{1}{2}\int_{x=0}^{x=5} f(u)\,du = \frac{1}{2}\int_0^{10} f(u)\,du = \frac{1}{2}(8) = \boxed{4}$

$f(2x)$ compresses $f(x)$ towards y-axis by a factor of $\frac{1}{2}$

So the Δx's = widths of rectangles are $\frac{1}{2}$ as wide.

And $f(2x)$ has all the wiggles between 0 & 5 that $f(x)$ has between 0 & 10.

16. Question Details SCalc8 4.5.060.

If f is continuous and $\int_0^4 f(x)\,dx = 12$, find $\int_0^2 x f(x^2)\,dx$.

$u = x^2 \Rightarrow du = 2x\,dx$

$u(0) = 0^2$

$u(2) = 2^2 = 4$

$= \frac{1}{2}\int_0^2 f(x^2)(2x\,dx) = \frac{1}{2}\int_{x=0}^{x=2} f(u)\,du$

$= \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}(12) = \boxed{6}$!

Nov 9-2:08 PM