Section 4.2 The Definite Integral

The Formal Definition

$f$ is cnt$^{s}$ on $[a,b]$

Then $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$, where

$x_k = a + k\Delta x$ and

$\Delta x = \frac{b-a}{n}$

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$\sum_{k=1}^{n} 1 = n = \underbrace{1+1+1+\cdots+1}_{n \text{ of 'em}}$

$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$ Gauss proved this (used it) at age 5.

$1+2+3+4+5+\cdots+97+98+99+100$

$(101)(50) = 101\left(\frac{100}{2}\right) = \frac{100(101)}{2}$

$\sum_{k=1}^{1000} k = \frac{1000(1001)}{2} = \frac{n(n+1)}{2}$

$= 500(1001)$ etc.

We Prove This.

$\mathbb{N} = \{1,2,3,\ldots\}$

Proof by induction

1. Show statement holds for $n=1$ $P(1)$
2. Show that any time it holds for $n=k$, it will hold for $k+1$, $P(k) \Rightarrow P(k+1)$

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Let $S$ = the set of all natural numbers $k$ such that the state $P(k)$ holds.

1. Show $1 \in S$ ($P(1)$ holds) (At least show $S \neq \emptyset$)
2. Show if $k \in S$, then $k+1 \in S$

Gauss's Result

Claim: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = P(n)$

Proof Let $S = \{n \mid n \in \mathbb{N} \text{ and } \sum_{k=1}^{n} k = \frac{n(n+1)}{2}\}$

$= \{n \mid n \in \mathbb{N} \text{ and } P(n) \text{ holds}\}$

Note $\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2} = 1$. So $1 \in S \neq \emptyset$.

Let $1 \leq n$ such that $P(n)$ holds. We WTS $P(n+1)$ holds.

$P(n): \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$P(n+1) = \sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$

$\sum_{k=1}^{n+1} k = \underbrace{1+2+3+\cdots+(n-1)+n}+(n+1)$

$= \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1)$

$= \frac{n^2+n+2(n+1)}{2} = \frac{n^2+n+2n+2}{2} = \frac{n^2+3n+2}{2}$

$= \frac{(n+1)(n+2)}{2} = P(n+1) \Rightarrow S = \mathbb{N}$ ■

**1.** Question Details SCalc8 4.2.001. [3395295]

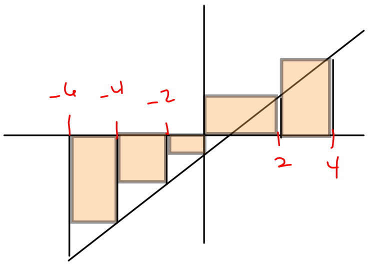
Evaluate the Riemann sum for $f(x) = 3x - 1$, $-6 \le x \le 4$, with five subintervals, taking the sample points to be right endpoints.

Explain, with the aid of a diagram, what the Riemann sum represents.

$\frac{4-(-6)}{5} = \frac{10}{5} = 2 = \Delta x$

$x_k = a + k\Delta x = -6 + 2k$

$k = 1, 2, 3, 4, 5$




The Riemann sum represents the SIGNED area "under" $3x-1$.

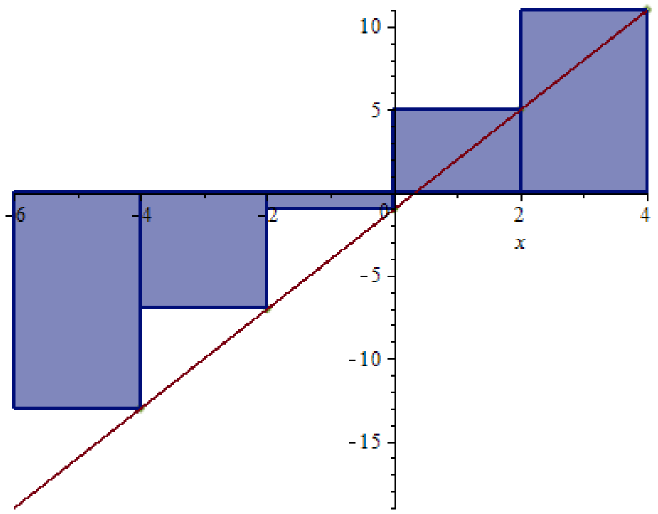
$3x - 1 = 0$

$3x = 1$

$x = \frac{1}{3}$

$$\Delta x \sum_{k=1}^{5} f(x_k) = 2\sum_{k=1}^{n}\left(3(-6+2k)-1\right) = 2\Big[\big(3(-6+2)-1\big) + \big(3(-6+4)-1\big) + \big(3(-6+6)-1\big) + \big(3(-6+8)-1\big) + \big(3(-6+10)-1\big)\Big]$$

$= 2[-13 - 7 - 1 + 5 + 11] = 2(-5) = -10$. Is the "net" area.



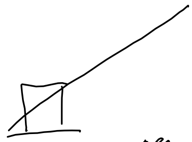

A right Riemann sum approximation of $\int_{-6}^{4} f(x)\,dx$, where $f(x) = 3x - 1$ and the partition is uniform. The approximate value of the integral is -10.00000000. Number of subintervals used: 5.

**2.** Question Details SCalc8 4.2.007.

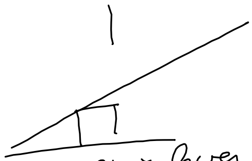
A table of values of an increasing function $f$ is shown. Use the table to find lower and upper estimates for

$$\int_{10}^{30} f(x)\,dx.$$

| $x$ | 10 | 14 | 18 | 22 | 26 | 30 |
|---|---|---|---|---|---|---|
| $f(x)$ | −15 | −8 | −3 | 2 | 5 | 7 |



Right is upper



Left is lower

$\Delta x = 4\ \forall.$

$R_5 = (14-10)(-8) + (18-14)(-3) + (22-18)(2) + (26-22)(5) + (30-26)(7)$

$= 4(-8) + 4(-3) + (4)(2) + 4(5) + 4(7)$

$= -32 - 12 + 8 + 20 + 28 = 12 = R = \text{upper}.$

$L_5 = 4[-15 - 8 - 3 + 2 + 5]$

$= 4[-19] = -76 = L_6 = \text{Lower}$

**3.** Question Details SCalc8 4.2.009. [3353909]

Use the Midpoint Rule with the given value of $n$ to approximate the integral. Round the answer to four decimal places.

$$\int_0^{80} \sin(\sqrt{x})\,dx, \quad n = 4$$

Midpoint Rule. Ugh.

$a + \Delta x$

$\Delta x = \frac{b-a}{n} = \frac{80}{n}$

$x_1 = 0 + \frac{1}{2}\left(\frac{80}{n}\right) = \frac{40}{n}$

$x_2 = \frac{40}{n} + \frac{80}{n} = \frac{120}{n}$

$x_3 = \frac{120}{n} + \frac{80}{n} = \frac{200}{n}$

$x_4 = \frac{280}{n}$

$x_1 = a + \frac{1}{2}\Delta x$

$x_2 = a + \frac{1}{2}\Delta x + \Delta x = a + \frac{3}{2}\Delta x$

$x_3 = a + \frac{5}{2}\Delta x$

$\vdots$

$x_n = a + \frac{2(n-1)+1}{2}\Delta x$

$x_2 = a + \frac{2(1)+1}{2}\Delta x$

$x_3 = a + \frac{2(2)+1}{2}\Delta x$

$$\frac{80}{n}\sum_{k=1}^{n} \sin\sqrt{\left(\frac{2(k-1)+1}{2}\cdot\frac{80}{n}\right)}$$

$= \frac{80}{n}\left(\sin\sqrt{40}\right.$

$n = 4$, dummy.

$= \frac{80}{4}\left[\sin\sqrt{\left(\frac{40}{4}\right)} + \sin\sqrt{\left(\frac{120}{4}\right)} + \sin\sqrt{\frac{200}{4}} + \sin\sqrt{\frac{280}{4}}\right]$

$= 20\left[\sin\sqrt{10} + \sin\sqrt{30} + \sin\sqrt{50} + \sin\sqrt{70}\right]$

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)+sin(√(70))
   -4.162626076
20*(sin(√(10))+s
in(√(30))+sin(√(
50))+sin(√(70)))

     16.76290818
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$\approx$ 16.76290818

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**4. Question Details** SCalc8 4.2.019

Express the limit as a definite integral on the given interval.

$$\lim_{n\to\infty}\sum_{i=1}^{n}[6(x_i^*)^3 - 9x_i^*]\Delta x, \quad [2, 6]$$

In 4.1, we had to deduce the interval in question from the delta-x, i.e., reverse-engineer the (b - a)/n to deduce b and a.

$$\int_2^6 (6x^3 - 9x)\,dx$$

$x_i^*$ means left, right or midpoint method, because, as $n\to\infty$, the rectangles are so skinny that left ≈ right ≈ midpoint

**5. Question Details** SCalc8 4.2.021.

Use the form of the definition of the integral given in the theorem to evaluate the integral.

$$\int_2^8 (4 - 2x)\,dx$$

"formal definition." Apparently these questions written by illiterates.

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left(4 - 2\left(2 + \frac{6k}{n}\right)\right)\left(\frac{6}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\left[4 - 2\left(2 + \frac{6(k-1)}{n}\right)\left(\frac{6}{n}\right)\right]$$

$$\Delta x = \frac{8-2}{n} = \frac{6}{n}$$

Right endpoints

$$x_k = 2 + k\Delta x = 2 + \frac{6k}{n}$$

Left: $x_k = 2 + (k-1)\Delta x = 2 + \frac{6(k-1)}{n}$

**6. Question Details** SCalc8 4.2.022.

Use the ~~form of the~~ definition of the integral given in the theorem to evaluate the integral.

$$\int_1^6 (x^2 - 4x + 7)\,dx$$

formal

$$\Delta x = \frac{b-a}{n} = \frac{6-1}{n} = \frac{5}{n}$$

$$x_k = 1 + k\Delta x = 1 + \frac{5k}{n}$$

Need fulThings

$$\sum_{k=1}^{n} 1 = n, \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Write much, think little.

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}(x_k^2 - 4x_k + 7)\Delta x$$

$$= \sum_{k=1}^{n}\left(\left(\frac{5k}{n}+1\right)^2 - 4\left(\frac{5k}{n}+1\right) + 7\right)\left(\frac{5}{n}\right)$$

$$= \frac{5}{n}\sum_{k=1}^{n}\left[\left(\frac{25k^2}{n^2} + \frac{10k}{n} + 1\right) - \frac{20k}{n} - 4 + 7\right]$$

$$= \frac{5}{n}\sum_{k=1}^{n}\left[\frac{25}{n^2}k^2 - \frac{10}{n}k + 4\right]$$

$$= \frac{5}{n}\sum_{k=1}^{n}\frac{25}{n^2}k^2 - \frac{5}{n}\sum_{k=1}^{n}\frac{10}{n}k + \frac{5}{n}\sum_{k=1}^{n}4$$

$$= \frac{5}{n}\cdot\frac{25}{n^2}\sum_{k=1}^{n}k^2 - \frac{5}{n}\cdot\frac{10}{n}\sum_{k=1}^{n}k + \frac{5}{n}\cdot 4\sum_{k=1}^{n}1$$

$$= \left(\frac{125}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{50}{n^2}\right)\left(\frac{n(n+1)}{2}\right) + \left(\frac{20}{n}\right)(n)$$

$$= \frac{125}{6n^3}(2n^3 + \dots) - \left(\frac{50}{2n^2}\right)(n^2 + \dots) + 20$$

$$\xrightarrow{n\to\infty} \frac{(125)(2n^3)}{6n^3} - \frac{50n^2}{2n^2} + 20$$

$$= \frac{125}{3} - 25 + 20 = \frac{125}{3} - \frac{5}{1}\cdot\frac{3}{3}$$

$$= \boxed{\frac{110}{3}}$$

① $\sum_{k=1}^{n}(a_k + b_k) = a_1 + b_1 + a_2 + b_2 + \dots + a_n + b_n$

$= a_1 + a_2 + \dots + a_n + b_1 + b_2 + \dots + b_n$

$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

② $\sum_{k=1}^{n} 5a_k = 5a_1 + 5a_2 + \dots + 5a_n$

$= 5[a_1 + a_2 + \dots + a_n]$

$= 5\sum_{k=1}^{n} a_k$

① & ② say that "$\sum$" is "linear"

(Linear: Respects sums & multiples by scalars (numbers)) Respects sums & scalar multiplication.)

$\frac{d}{dx}$ is like that

$$\frac{d}{dx}[af(x) + bg(x)] = a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)]$$

$$\sum[ab_k + cd_k] = a\sum b_k + c\sum d_k$$

**7.** Question Details SCalc8 4.2.026. [33540

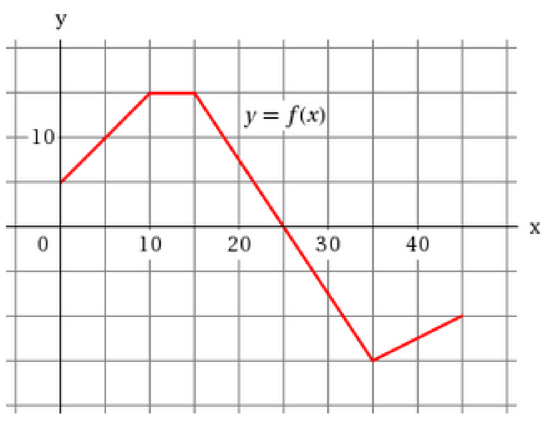
(a) Find an approximation to the integral $\int_0^4 (x^2 - 2x)\,dx$ using a Riemann sum with right endpoints and $n = 8$.

(b) If $f$ is integrable on $[a, b]$, then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\,\Delta x$. Use this to evaluate $\int_0^4 (x^2 - 2x)\,dx$.

i.e., evaluate the integral by the limit definition.

**8.** Question Details SCalc8 4.2.033.

The graph of $f$ is shown. Evaluate each integral by interpreting it in terms of areas.




(a) $\int_0^{10} f(x)\,dx$

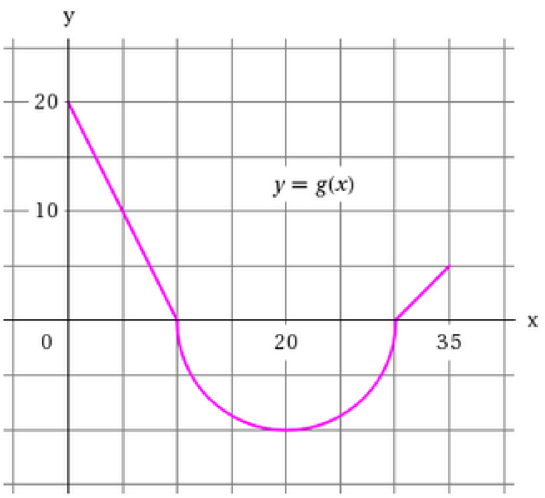
(b) $\int_0^{25} f(x)\,dx$

(c) $\int_{25}^{35} f(x)\,dx$

(d) $\int_0^{45} f(x)\,dx$

**9.** Question Details SCalc8 4.2.034.MI.

The graph of $g$ consists of two straight lines and a semicircle. Use it to evaluate each integral.




(a) $\int_{0}^{10} g(x)\,dx$

(b) $\int_{10}^{30} g(x)\,dx$

(c) $\int_{0}^{35} g(x)\,dx$

**10.** Question Details SCalc8 4.2.037.

Evaluate the integral by interpreting it in terms of areas.

$$\int_{-10}^{0} \left(3 + \sqrt{100 - x^2}\right) dx$$

**11.** Question Details SCalc8 4.2.043.

Given that $\int_0^1 x^2\,dx = \frac{1}{3}$, use this fact and the properties of integrals to evaluate $\int_0^1 (5 - 3x^2)\,dx$.

**12.** Question Details SCalc8 4.2.046. [3353875]

Given that $\int_a^b x\,dx = \frac{b^2 - a^2}{2}$, use this result and the fact that $\int_0^{\pi/2} \cos(x)\,dx = 1$, together with the properties of integrals, to evaluate $\int_0^{\pi/2} (2\cos(x) - 3x)\,dx$.

**13.** Question Details SCalc8 4.2.047.

Write as a single integral in the form $\int_a^b f(x)\,dx$.

$$\int_{-3}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-3}^{-1} f(x)\,dx$$

**14.** Question Details SCalc8 4.2.052.

If $F(x) = \int_2^x f(t)\,dt$, where $f$ is the function whose graph is given, which of the following values is largest?

- ○ $F(0)$
- ○ $F(1)$
- ○ $F(2)$
- ○ $F(3)$
- ○ $F(4)$

**15.** Question Details SCalc8 4.2.054.

Suppose $f$ has absolute minimum value $m$ and absolute maximum value $M$. Between what two values must $\int_2^5 f(x)\,dx$ lie?

Which property of integrals allows you to make your conclusion?

**16.** Question Details SCalc8 4.2.057. [3353944

Use the properties of integrals to choose the inequality that would make the statement true without evaluating the integrals.

$12 \ [\le] \ \int_{-3}^{3} \sqrt{4 + x^2}\,dx \ [\le] \ 6\sqrt{13}$