Section 4.3 - Definite Integrals, the Hard Way (by the definition).

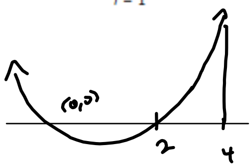
#7

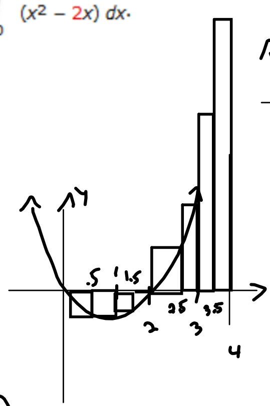
(a) Find an approximation to the integral $\int_0^4 (x^2 - 2x)\,dx$ using a Riemann sum with right endpoints and $n = 8$.

(b) If f is integrable on $[a, b]$, then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\,\Delta x$. Use this to evaluate $\int_0^4 (x^2 - 2x)\,dx$.

$x^2 - 2x = x(x-2) = 0 \Rightarrow$

$x = 0, 2.$





$\Delta x = .5$ by picture

$\frac{b-a}{n} = \frac{4-0}{8} = \frac{1}{2} = .5$ ✓

$x_k = a + k\Delta x = 0 + k \cdot \frac{1}{2} = \frac{1}{2}k = \frac{k}{2}$

$f(x) = x^2 - 2x$

$f(x_k) = x_k^2 - 2x_k$

(Signed)

$$\text{Area} \approx \Delta x \sum_{k=1}^{n} f(x_k) = \frac{1}{2}\sum_{k=1}^{8}(x_k^2 - 2x_k) = \frac{1}{2}\sum_{k=1}^{8}\left(\left(\frac{k}{2}\right)^2 - 2\left(\frac{k}{2}\right)\right)$$

$$= \frac{1}{2}\sum_{k=1}^{8}\left(\frac{k^2}{4} - k\right) = \frac{1}{2}\cdot\frac{1}{4}\sum_{k=1}^{8}k^2 - \frac{1}{2}\sum_{k=1}^{8}k$$

$$= \frac{1}{8}\left(\frac{8(9)(2(8)+1)}{6}\right) - \frac{1}{2}\left(\frac{8(9)}{2}\right) = \frac{1}{48}(8)(9)(17) - \frac{1}{2}(4)(9)$$

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$(n = 8)$

$$= \frac{1}{6}(9)(17) - (2)(9)$$

$$= \frac{1}{2}(3)(17) - 18$$

$$= \frac{51}{2} - \frac{36}{2} = \boxed{\frac{15}{2}} \approx \text{Area.}$$

(b) $\Delta x = \frac{b-a}{n} = \frac{4}{n}$

$x_k = a + k\Delta x = 0 + k \cdot \frac{4}{n} = \frac{4k}{n}$

$$\Delta x \sum_{k=1}^{n} f(x_k) = \frac{4}{n}\sum_{k=1}^{n}(x_k^2 - 2x_k) = \frac{4}{n}\sum_{k=1}^{n}\left(\left(\frac{4k}{n}\right)^2 - 2\left(\frac{4k}{n}\right)\right)$$

$$= \frac{4}{n}\sum_{k=1}^{n}\left(\frac{16k^2}{n^2} - \frac{8k}{n}\right) = \frac{4}{n}\sum_{k=1}^{n}\frac{16}{n^2}k - \frac{4}{n}\sum_{k=1}^{n}\frac{8}{n}k$$

$$= \frac{4}{n}\cdot\frac{16}{n^2}\left(\frac{n^3 + \cdots}{3}\right) - \frac{4}{n}\cdot\frac{8}{n}\left(\frac{n^2 + \cdots}{2}\right)$$

$$\xrightarrow{n\to\infty} \frac{16\cdot 4}{3} - 16 = \frac{64-48}{3} = \boxed{\frac{14}{3}}$$

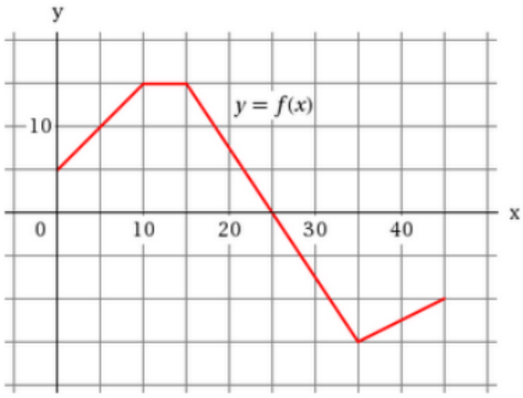
§4.3 Check:

FTC II

$$\int_0^4 (x^2 - 2x)\,dx = \left.\frac{x^3}{3} - x^2\right]_0^4 = \frac{64}{3} - 16 = \frac{64-48}{3} = \frac{14}{3}$$ ✓

#8

The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.



(a) $\int_0^{10} f(x)\,dx$

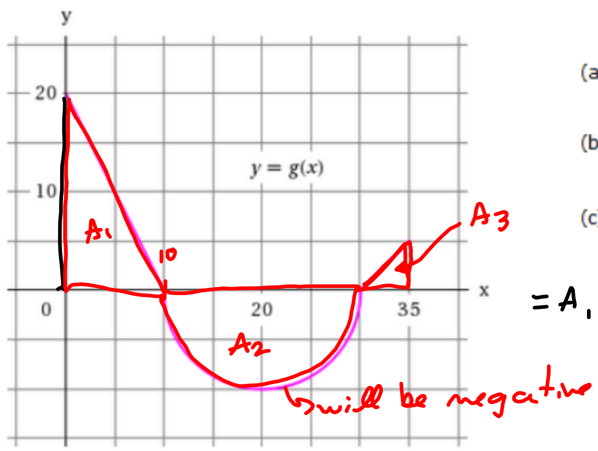
(b) $\int_0^{25} f(x)\,dx$

(c) $\int_{25}^{35} f(x)\,dx$

(d) $\int_0^{45} f(x)\,dx$

#9

The graph of g consists of two straight lines and a semicircle. Use it to evaluate each integral.



(a) $\int_0^{10} g(x)\,dx = A_1 = \frac{1}{2}bh = \frac{1}{2}(10)(20)$
$\boxed{=100} = A_1$

(b) $\int_{10}^{30} g(x)\,dx = A_2 = -\frac{1}{2}(\pi(10)^2) = \boxed{-50\pi}$
$= A_2$
-50π!

(c) $\int_0^{35} g(x)\,dx$

$= A_1 + A_2 + A_3$

$= 100 - 50\pi + \frac{1}{2}(5)(5)$

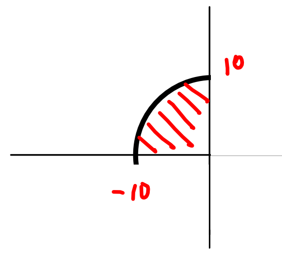
$= 100 - 50\pi + \frac{25}{2}$

$= \dfrac{225 - 100\pi}{2}$

#10

Evaluate the integral by interpreting it in terms of areas.

$$\int_{-10}^{0} \left(3 + \sqrt{100 - x^2}\right) dx$$



$\frac{1}{4}$ area of circle

$$\int_{-10}^{0} \sqrt{100-x^2}\, dx$$

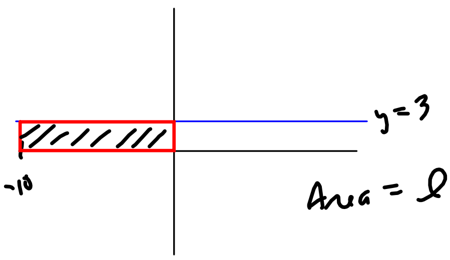
$$= \frac{1}{4}\pi(10)^2 = 25\pi$$

$x^2 + y^2 = 100$

$\Rightarrow y^2 = 100 - x^2$

$\Rightarrow y = \pm\sqrt{100-x^2}$

$y = \sqrt{100-x^2}$ is top $\frac{1}{2}$ of a circle of radius $r = 10$ centered @ $(0,0)$



Area $= l \cdot h = 10 \cdot 3 = 30$

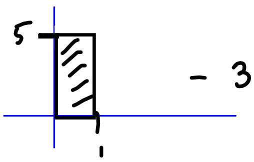
$$\int_{-10}^{0} \left(3+\sqrt{100-x^2}\right)dx = \int_{-10}^{0} 3\,dx + \int_{-10}^{0} \sqrt{100-x^2}\,dx$$

$$= \boxed{30 + 25\pi}$$

#11

Given that $\int_0^1 x^2\,dx = \frac{1}{3}$, use this fact and the properties of integrals to evaluate $\int_0^1 (5 - 3x^2)\,dx$.

$$\int_0^1 (5-3x^2)\,dx = \int_0^1 5\,dx - 3\int_0^1 x^2\,dx =$$

$=$ 

$$-3\left[\tfrac{1}{3}\right] = 5(1) - 1 = \boxed{4}$$

#12

Given that $\int_a^b x\,dx = \frac{b^2 - a^2}{2}$, use this result and the fact that $\int_0^{\pi/2} \cos(x)\,dx = 1$, together with the properties of integrals, to evaluate $\int_0^{\pi/2} (2\cos(x) - 3x)\,dx$.

$$= 2\int_0^{\frac{\pi}{2}} \cos(x)\,dx - 3\int_0^{\frac{\pi}{2}} x\,dx = 2(1) - 3\left(\frac{\left(\frac{\pi}{2}\right)^2 - 0^2}{2}\right) = 2 - \frac{3}{2}\left(\frac{\pi^2}{4}\right)$$

$$= 2 - \frac{3}{8}\pi^2 = \boxed{\frac{16 - 3\pi^2}{8}}$$

#13

Write as a single integral in the form $\int_a^b f(x)\,dx$.

$$\int_{-3}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-3}^{-1} f(x)\,dx$$

$$= \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-3}^{-1} f(x)\,dx$$

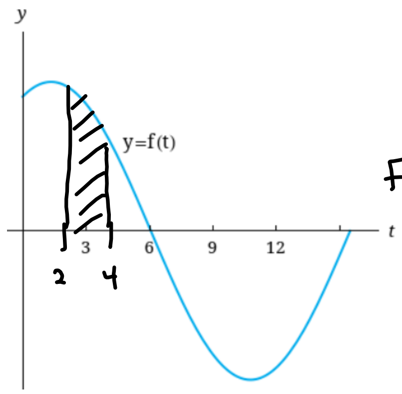
$$= \boxed{\int_{-1}^{5} f(x)\,dx}$$

#14

If $F(x) = \int_2^x f(t)\,dt$, where f is the function whose graph is given, which of the following values is largest?

- ○ $F(0)$
- ○ $F(1)$
- ○ $F(2)$
- ○ $F(3)$
- ○ $F(4)$

$F(6) > F(4)$



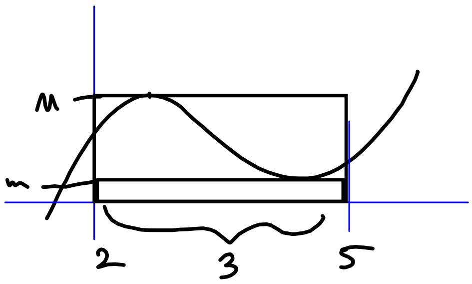
As long as $f(t) > 0$, the definite integral is an increasing function.

$F(x) = \int_2^x f(t)$ is increasing function of x on $(0, 6)$

#15

Suppose f has absolute minimum value m and absolute maximum value M. Between what two values must $\int_2^5 f(x)\,dx$ lie?

Which property of integrals allows you to make your conclusion?



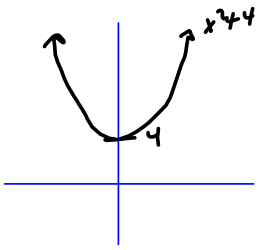
$$m(5-2) \le \int_2^5 f(x)\,dx \le M\cdot 3$$

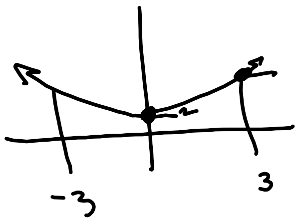
$$3m \le \int_2^5 f(x)\,dx \le 3M$$

#16

Use the properties of integrals to choose the inequality that would make the statement true without evaluating the integrals.

12 [≤] $\int_{-3}^{3} \sqrt{4+x^2}\,dx$ [≤] $6\sqrt{13}$





$$\sqrt{3^2+4} = \sqrt{9+4} = \sqrt{13}$$

$$2 \le \sqrt{x^2+4} \le \sqrt{13}$$

$$\int_{-3}^{3} 2 \le \int_{-3}^{3} \sqrt{x^2+4} \le \int_{-3}^{3} \sqrt{13}$$

$$\int_{-3}^{3} 2 = 2(3--3) = 2(6) = 12 \qquad \int_{-3}^{3} \sqrt{13} = 6\sqrt{13}$$

$$12 \le \int_{-3}^{3} \sqrt{x^2+4}\,dx \le 6\sqrt{13}$$