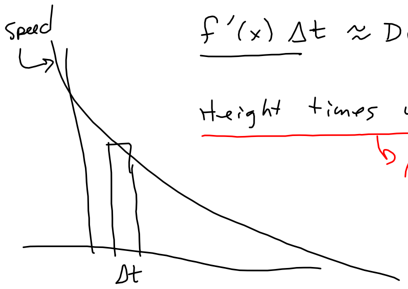
Section 4.1 Distances and Areas

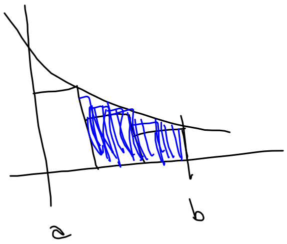
Rate times Time = Distance.

$f'(x)\,\Delta t \approx$ Distance

Height times width = Area = Distance.

Area under the curve





Sum of the rectangles approximates the area under the curve.

more rectangles $\Longrightarrow$ Better Estimate.

Take # of rectangles to ∞ & find exact area.

Take limit as the widest rectangle's width goes to zero.

That still makes an ∞ # of rectangles.

In this Book, we generally make all the widths the same. Then # rectangles $\longrightarrow \infty$ is identical to

width of rectangles $\longrightarrow 0$

Width of WIDEST rectangle is called the "mesh of the partition."

For us, the mesh is $\frac{b-a}{n}$, where $n =$ the # of rectangles

$[a,b] = [0,10]$, $n = 50$, then mesh $= \Delta x = \frac{b-a}{n} = \frac{10}{50} = \frac{1}{5}$

Height of the rectangles is $f(x)$.

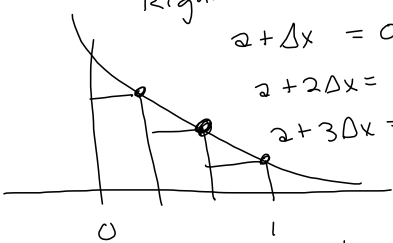
$$\sum_{k=1}^{n} f(x_k)\,\Delta x$$

Right endpoints

$a+\Delta x = 0 + \frac{1}{3} = \frac{1}{3} = x_1$

$a+2\Delta x = 0+2\left(\frac{1}{3}\right) = \frac{2}{3} = x_2$

$a+3\Delta x = 0+3\left(\frac{1}{3}\right) = 1 = x_3 = a+3\Delta x$

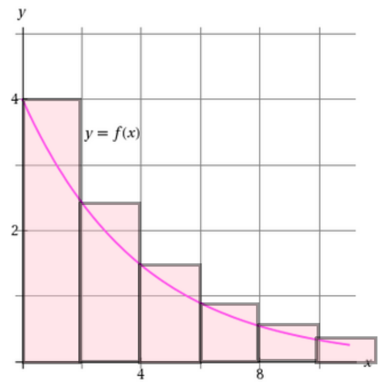


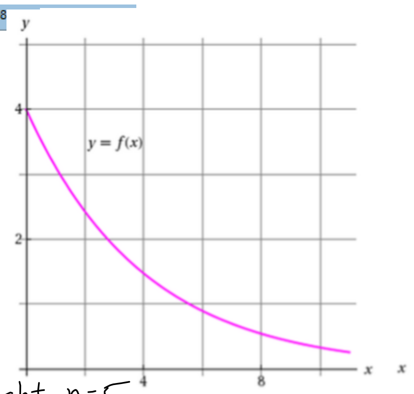
$$\text{Area} \approx \left(f\left(\tfrac{1}{3}\right)\right)\left(\tfrac{1}{3}\right) + \left(f\left(\tfrac{2}{3}\right)\right)\left(\tfrac{1}{3}\right) + \left(f(1)\right)\left(\tfrac{1}{3}\right)$$

$$= \sum_{k=1}^{3} \left(f\left(\tfrac{k}{3}\right)\right)\left(\tfrac{1}{3}\right) = \sum_{k=1}^{3} f(x_k)\,\Delta x$$

1. Question Details SCalc8

Consider the following.





Right, $n=5$

$$\text{Area} \approx \sum f(x_k)\Delta x = (2.4)(1)+(1.4)(1)+(.8)(1)+.(5)(1)+(.4)(1)$$

$$=(2.4+1.4+.8+.5+.4)(1)$$

$$=5.5$$

(a) By reading values from the given graph of f, use five rectangles to find a lower estimate for the area under the given graph of f from $x = 0$ to $x = 10$. (Round your answer to one decimal place.)

Sketch the rectangles that you use.

By reading values from the given graph of f, use five rectangles to find an upper estimate for the area under the given graph of f from $x = 0$ to $x = 10$. (Round your answer to one decimal place.)

Sketch the rectangles that you use.

(b) Find new estimates using ten rectangles in each case. (Round your answers to one decimal place.)

$$\sum f(x_k)\,\Delta x = 4+2.4+1.5+.9+.7+.4$$

$= 6.6$ is an upper estimate

(b) No fun. No real insight.

2. Question Details SCalc8 4.1.005. [335

(a) Estimate the area under the graph of $f(x) = 3 + 2x^2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints.

Then improve your estimate by using six rectangles.

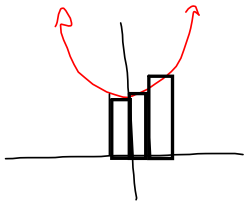
Sketch the curve and the approximating rectangles for R_3.

Sketch the curve and the approximating rectangles for R_6.

(b) Repeat part (a) using left endpoints.

(c) Repeat part (a) using midpoints.

(d) From your sketches in parts (a)-(c), which appears to be the best estimate?



$\Delta x = \frac{b-a}{3} = \frac{2-(-1)}{3} = \frac{3}{3} = 1$

Right:

$x_1 = a + \Delta x = -1 + 1 = 0$

$x_2 = x_1 + \Delta x = 0 + 1 = 1$

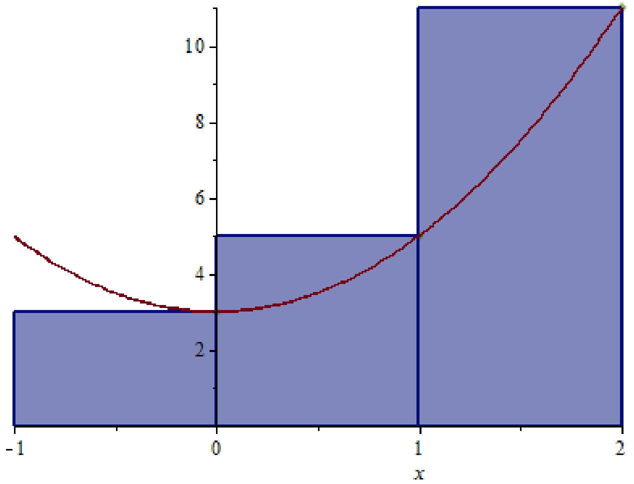
$x_3 = x_2 + \Delta x = 1 + 1 = 2$

$f(x_1) = f(0) = 2(0)^2 + 3 = 3$

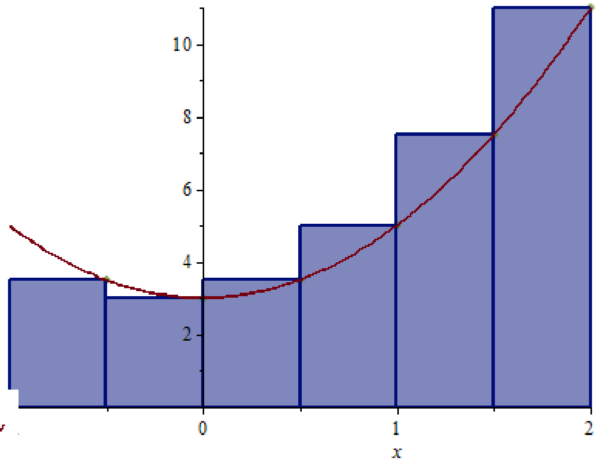
$f(x_2) = f(1) = 2(1)^2 + 3 = 5$

$f(x_3) = f(2) = 2(2)^2 + 3 = 11$

$\sum_{k=1}^{3} f(x_k)\Delta x = 3\cdot 1 + 5\cdot 1 + 11\cdot 1 = (3+5+11)(1) = \boxed{19}$

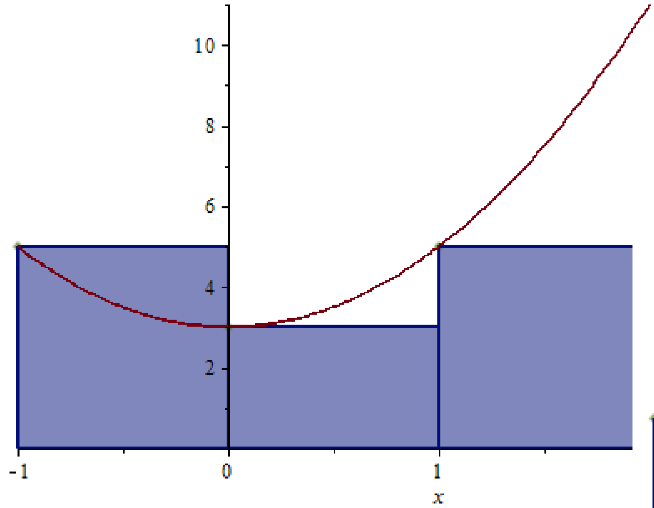


A right Riemann sum approximation of $\int_{-1}^{2} f(x)\,dx$, where $f(x) = 2x^2 + 3$ and the partition is uniform. The approximate value of the integral is 19.00000000. Number of subintervals used: 3.

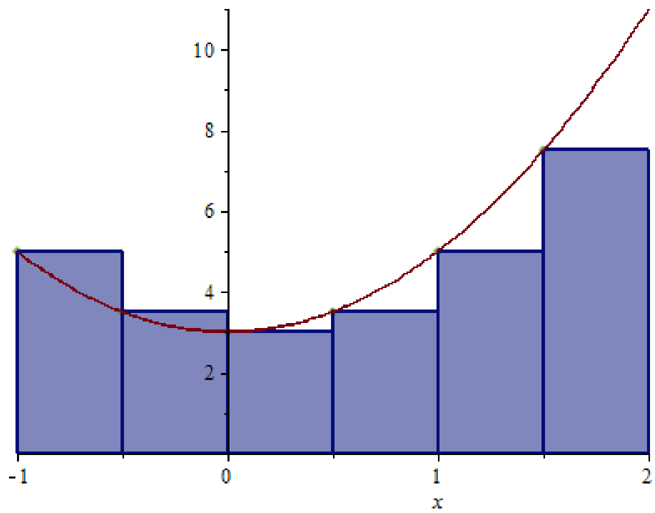


A right Riemann sum approximation of $\int_{-1}^{2} f(x)\,dx$, where

c) $= 2x^2 + 3$ and the partition is uniform. The approximate value of



A left Riemann sum approximation of $\int_{-1}^{2} f(x)\,dx$, where $f(x) = 2x^2 + 3$ and the partition is uniform. The approximate value of the integral is 13.00000000. Number of subintervals used: 3.



A left Riemann sum approximation of $\int_{-1}^{2} f(x)\,dx$, where $f(x) = 2x^2 + 3$ and the partition is uniform. The approximate value of the integral is 13.75000000. Number of subintervals used: 6.

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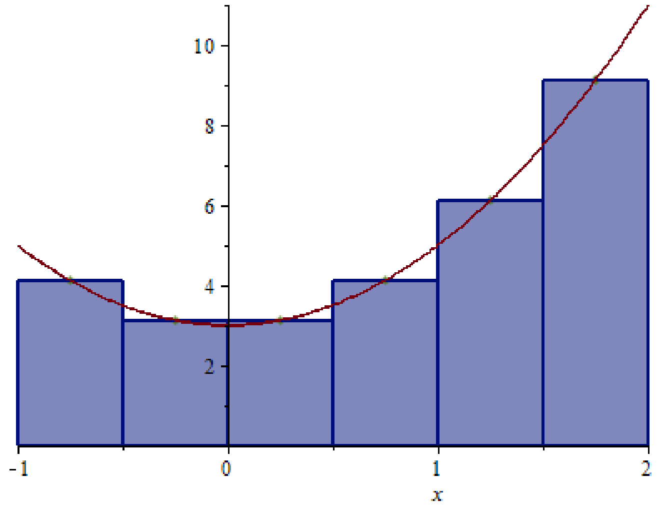
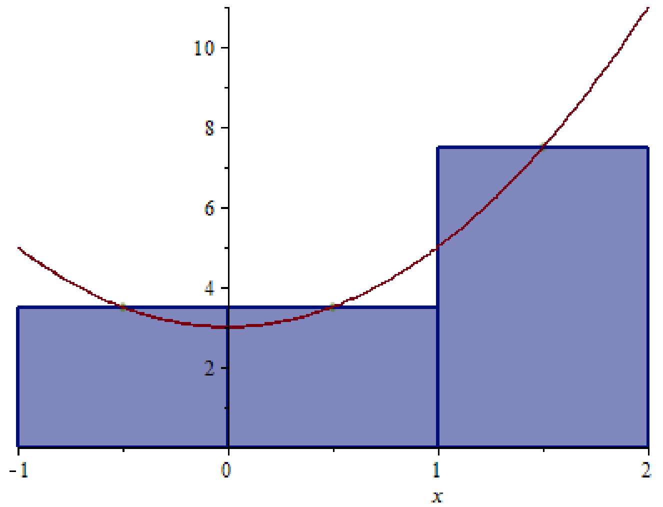


A midpoint Riemann sum approximation of $\int_{-1}^{2} f(x)\,dx$, where $f(x) = 2x^2 + 3$ and the partition is uniform. The approximate value of the integral is 14.87500000. Number of subintervals used: 6.



A midpoint Riemann sum approximation of $\int_{-1}^{2} f(x)\,dx$, where $f(x) = 2x^2 + 3$ and the partition is uniform. The approximate value of the integral is 14.50000000. Number of subintervals used: 3.

Midpoints seems to be more accurate.

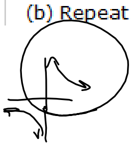
3. Question Details SCalc8 4.1.003. [3395143]

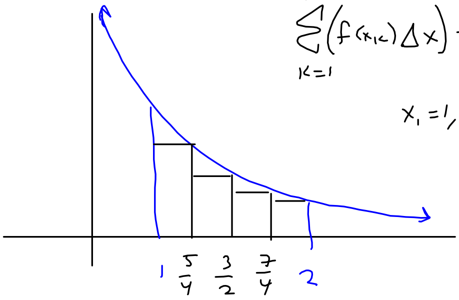
(a) Estimate the area under the graph of $f(x) = 3/x$ from $x = 1$ to $x = 2$ using four approximating rectangles and right endpoints. (Round your answer to four decimal places.)

Is your estimate an underestimate or an overestimate?

(b) Repeat part (a) using left endpoints. (Round your answer to four decimal places.)

$\frac{3}{x}$ is decreasing. Right Endpoints are an under-estimate.





$$\sum_{k=1}^{4}\left(f(x_k)\Delta x\right) = \left(\sum_{k=1}^{4} f(x_k)\right)\Delta x$$

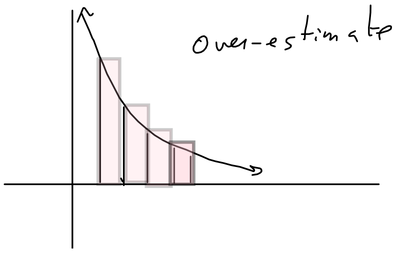
$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

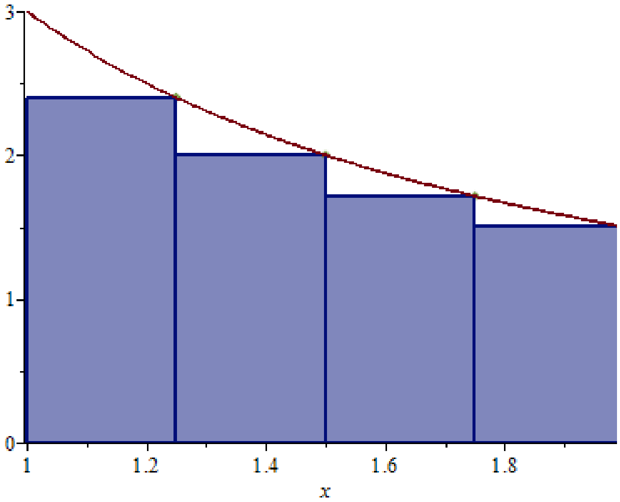
$x_1 = 1,\ x_2 = 1 + \frac{1}{4} = \frac{5}{4},\ x_3 = \frac{5}{4} + \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$

Left:

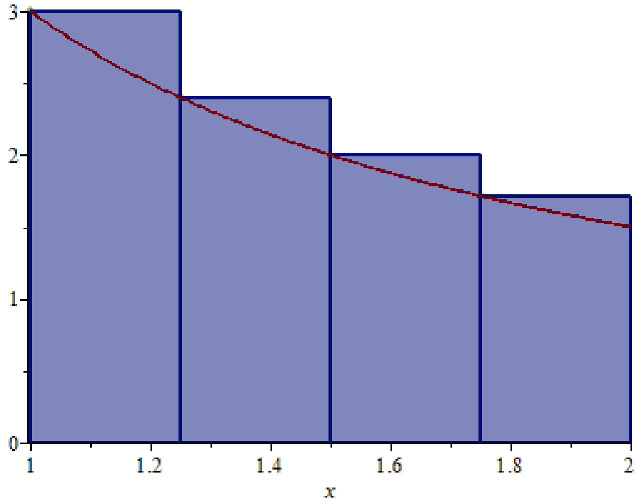
$x_1 = 1,\ x_2 = \frac{5}{4},\ x_3 = \frac{3}{2},\ x_4 = \frac{7}{4}$

Over-estimate





A right Riemann sum approximation of $\int_1^2 f(x)\,dx$, where $f(x) = \frac{3}{x}$ and the partition is uniform. The approximate value of the integral is 1.903571428. Number of subintervals used: 4.



A left Riemann sum approximation of $\int_1^2 f(x)\,dx$, where $f(x) = \frac{3}{x}$ and the partition is uniform. The approximate value of the integral is 2.278571428. Number of subintervals used: 4.

4. Question Details SCalc8 4.1.010. [3353747]

With a programmable calculator (or a computer), it is possible to evaluate the expressions for the sums of areas of approximating rectangles, even for large values of n, using looping. (On a TI use the Is> command or a For-EndFor loop, on a Casio use Isz, on an HP or in BASIC use a FOR-NEXT loop.) Compute the sum of the areas of approximating rectangles using equal subintervals and right end points for $n = 10, 30, 50$, and 100. (Round your answers to four decimal places.)

The region under $y = 2\cos(x)$ from 0 to $\pi/2$

n	Sum of Areas
10	2.1530
30	
50	1.9684
100	

Guess the value of the exact area.

$[a,b] = [0, \frac{\pi}{2}]$

$n=10: \quad \frac{\frac{\pi}{2}-0}{10} = \frac{\pi}{20} = \Delta x = \frac{b-a}{n}$

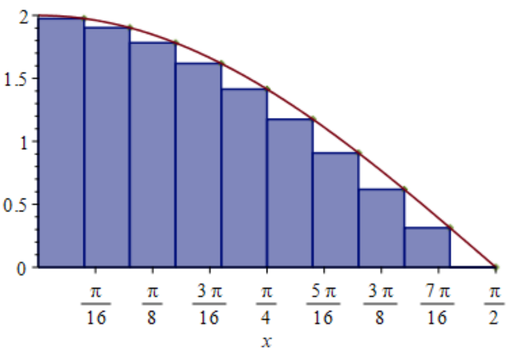
$\sum_{k=1}^{10} 2\cos\left(\frac{\pi}{20}k\right)\cdot\frac{\pi}{20}$

$\sum_{k=1}^{30} 2\cos\left(\frac{\pi}{60}k\right)\cdot\frac{\pi}{60} = 2\left[\cos\left(\frac{\pi}{60}\right) + \cos\left(\frac{2\pi}{60}\right) + \dots + \cos\left(\frac{\pi}{2}\right)\right]\left(\frac{\pi}{60}\right)$

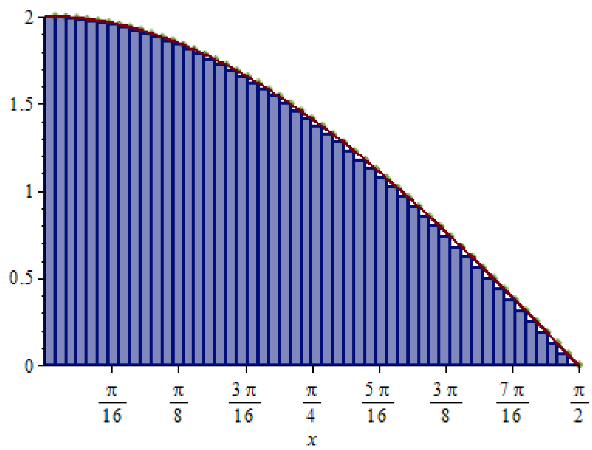
$n=50: \quad \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{50} = \frac{\pi}{100}$

$\sum_{k=1}^{50} 2\cos\left(\frac{\pi}{100}k\right)\cdot\frac{\pi}{100}$

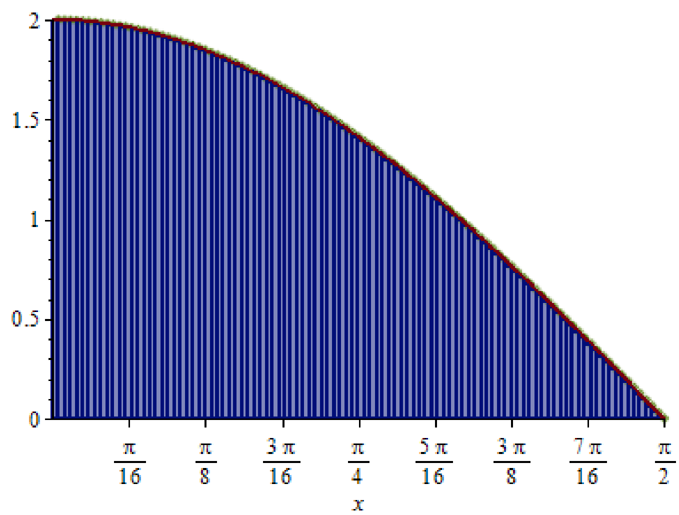
$\sum_{k=1}^{100} 2\cos\left(\frac{\pi}{200}k\right)\cdot\frac{\pi}{200}$



A right Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x)\,dx$, where $f(x) = 2\cos(x)$ and the partition is uniform. The approximate value of the integral is 1.838806340. Number of subintervals used: 10.

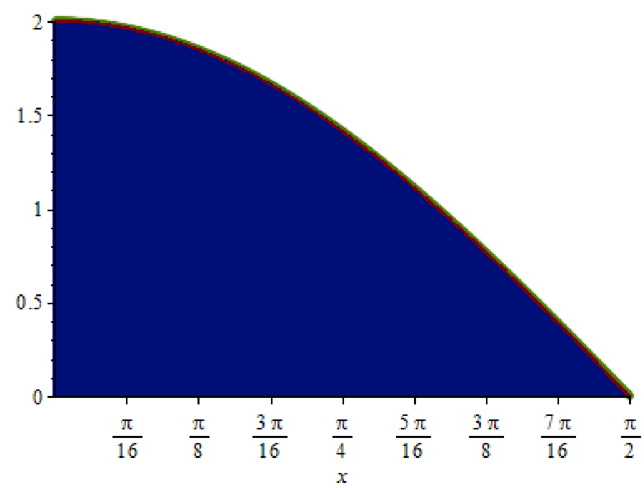


A right Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x)\,dx$, where $f(x) = 2\cos(x)$ and the partition is uniform. The approximate value of the integral is 1.968419577. Number of subintervals used: 50.



A right Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x)\,dx$, where $f(x) = 2\cos(x)$ and the partition is uniform. The approximate value of the integral is 1.984250914. Number of subintervals used: 100.

$S \approx 2.0000$?



A right Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x)\,dx$, where $f(x) = 2\cos(x)$ and the partition is uniform. The approximate value of the integral is 1.998428792. Number of subintervals used: 1000.

$n = 1000$

5. **Question Details** SCalc8 4.1.015. [3353805]

Oil leaked from a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed, and values of the rate at two hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.

t (h)	0	2	4	6	8	10
$r(t)$ (L/h)	8.7	7.6	6.8	6.4	5.7	5.3

$\Delta x = 2$

Decreasing.

Lower estimate @ right endpoints

$(7.6 + 6.8 + 6.4 + 5.7 + 5.3)(2)$

$= (31.8)(2) = 63.6$

Upper on left endpoints

$(8.7 + 7.6 + 6.8 + 6.4 + 5.7)(2)$

$(35.5)(2) = 71$

6. Question Details SCalc8 4.1.016.MI. [3353883]

When we estimate distances from velocity data, it is sometimes necessary to use times $t_0, t_1, t_2, t_3, \ldots$ that are not equally spaced. We can still estimate distances using the time periods $\Delta t_i = t_i - t_{i-1}$. For example, a space shuttle was launched on a mission, the purpose of which was to install a new motor in a satellite. The table provided gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters. Use these data to estimate the height, h, above Earth's surface of the space shuttle, 62 seconds after liftoff. (Give the upper approximation available from the data.)

$h =$ ____ ft

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	180
End roll maneuver	15	319
Throttle to 89%	20	442
Throttle to 67%	32	742
Throttle to 104%	59	1217
Maximum dynamic pressure	62	1453
Solid rocket booster separation	125	4151

Increasing values

upper estimate is right end points.

Δx's aren't same width!

$(180)(10) + (319)(5) + (442)(5)$

$+ (742)(12) + (1217)(27) + (1453)(3)$

$+ \ (4151)(63) =$ ____

7. Question Details SCalc8 4.1.021.

Use the definition to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

$f(x) = \dfrac{8x}{x^2 + 7}, \quad 1 \le x \le 3 \rightarrow \Delta x = \dfrac{b-a}{n} = \dfrac{3-1}{n} = \dfrac{2}{n}$

Right Endpoints: $x_k = a + k \cdot \Delta x = 1 + \dfrac{2k}{n}$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} \frac{8x_k}{x_k^2 + 7} \cdot \frac{2}{n} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{8\left(\frac{2k}{n}+1\right)}{\left(\frac{2k}{n}+1\right)^2 + 7} \right)\left(\frac{2}{n}\right)$$

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8. Question Details SCalc8 4.1.022. [3

Use the definition to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

$f(x) = x^2 + \sqrt{1 + 2x}$, $2 \le x \le 4$

$\Delta x = \frac{b-a}{n} = \frac{4-2}{n} = \frac{2}{n}$

$x_k = a + k\Delta x = 2 + \frac{2k}{n}$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\left(2 + \frac{2k}{n}\right)^2 + \sqrt{1 + 2\left(2 + \frac{2k}{n}\right)} \right)$$

9. Question Details SCalc8 4.1.024.

Determine a region whose area is equal to the given limit. Do not evaluate the limit.

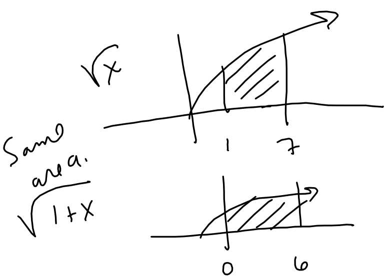
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n}\sqrt{1 + \frac{6i}{n}}$$

Jeopardy!

$\Delta x = \frac{6}{n}$, so width is 6.

$f(x) = \sqrt{x}$, on $[1, 7]$

$f(x) = \sqrt{1+x}$ on $[0, 6]$



10. Question Details SCalc8 4.1.025

Determine a region whose area is equal to the given limit. Do not evaluate the limit.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{8n} \tan\left(\frac{i\pi}{8n}\right)$$

$x_k = \frac{\pi}{8n} k = a + \frac{b-a}{n} k$

$a = 0$, $\Delta x = \frac{\pi}{8n} = \frac{\frac{\pi}{8}}{n}$

Area under $\tan(x)$ from $x=0$ to $x = \frac{\pi}{8}$

11. Question Details SCalc8 4.1.027. [3354057]

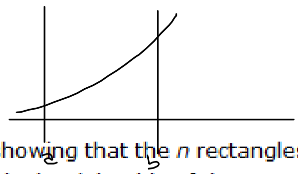
Let A be the area under the graph of an increasing continuous function f from a to b, and let L_n and R_n be the approximations to A with n subintervals using left and right endpoints, respectively.

(a) How are A, L_n, and R_n related?

$L_n < A < R_n$

(b) Show that

$$R_n - L_n = \frac{b-a}{n}[f(b) - f(a)]$$



Then draw a diagram to illustrate this equation by showing that the n rectangles representing $R_n - L_n$ can be reassembled to form a single rectangle whose area is the right side of the equation.

Nah

(c) Deduce that

$$R_n - A < \frac{b-a}{n}[f(b) - f(a)].$$

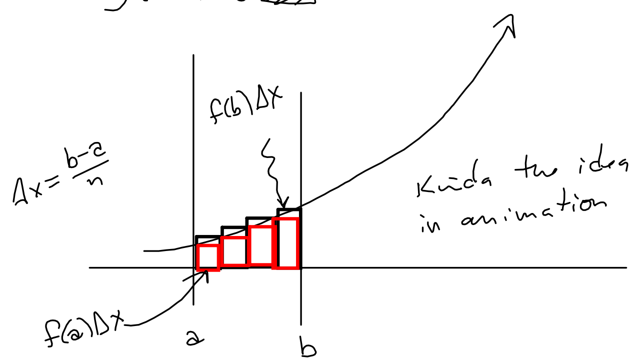
$$R_n - L_n = \sum_{k=1}^{n} f(a+k\Delta x)\Delta x - \sum_{k=1}^{n} f(a+(k-1)\Delta x)\Delta x$$

$$f(a+\Delta x)\Delta x + f(a+2\Delta x)\Delta x + \cdots + f(a+(n-1)\Delta x)\Delta x + f(a+n\Delta x)\Delta x$$

$$- f(a)\Delta x - f(a+\Delta x)\Delta x - f(a+2\Delta x)\Delta x - \cdots - f(a+(n-2)\Delta x)\Delta x - f(a+(n-1)\Delta x)\Delta x$$

$$= f(a+n\Delta x)\Delta x - f(a)\Delta x = \left(f(a+n\Delta x) - f(a)\right)\frac{b-a}{n}$$

$$= \left(f(b) - f(a)\right)\left(\frac{b-a}{n}\right)$$



(c) $R_n - A < \frac{b-a}{n}[f(b) - f(a)]$

$10 - 5 > 10 - 6$

$R_n - L_n > R_n - A$

$L_n < A \implies$

$-L_n > -A \implies$

$\frac{b-a}{n}[f(b) - f(a)] = R_n - L_n > R_n - A$

12. Question Details SCalc8 4.1.501.XP. [3389947]

The area A of the region S that lies under the graph of the continuous function is the limit of the sum of the areas of approximating rectangles.

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x]$$

Use this definition to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

$f(x) = \sqrt[3]{x}$, $1 \le x \le 13$

Right Endpoints

$$\frac{b-a}{n} = \Delta x = \frac{13-1}{n} = \frac{12}{n}$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n \to \infty} \sum_{k=1}^{n} \left(\sqrt[3]{1 + \frac{12k}{n}}\right)\left(\frac{12}{n}\right)$$

$$x_k = a + k\Delta x$$

$$= 1 + \frac{12k}{n}$$