1. Question Details S

Consider the following problem: Find two numbers whose sum is 23 and whose product is a maximum.

$x + y = 23 \Rightarrow y = 23 - x$

$P = xy$ to be maximized

$= x(23 - x) = 23x - x^2$

$\Rightarrow \frac{dP}{dx} = 23 - 2x \overset{SET}{=} 0$

$\Rightarrow 2x = 23$

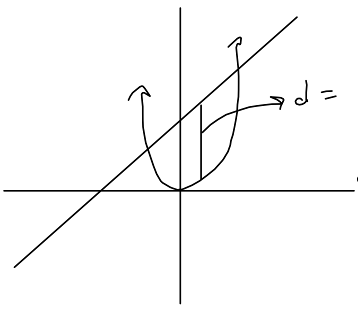
$x = \frac{23}{2}$

$23 - x = 23 - \frac{23}{2} = \frac{23}{2}$

$\boxed{x = y = \frac{23}{2}}$

2. Question Details SCalc8 3

What is the maximum vertical distance between the line $y = x + 6$ and the parabola $y = x^2$ for $-2 \le x \le 3$?



$d = x + 6 - x^2 = -x^2 + x + 6$

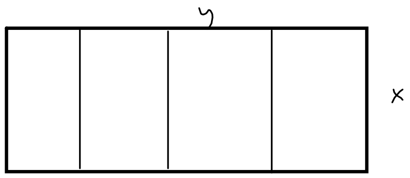
$\frac{d}{dx}[d] = -2x + 1 \overset{SET}{=} 0$

$\Rightarrow x = \frac{1}{2}$

$d\left(\frac{1}{2}\right) = \frac{1}{2} + 6 - \left(\frac{1}{2}\right)^2 = \frac{2 + 24 - 1}{4} = \boxed{\frac{25}{4} = d}$

3. Question Details SCalc8 3.7.011. [3353780]

Consider the following problem: A farmer with 850 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?



$5x + 2y = 850 \Rightarrow 2y = 850 - 5x$

$y = \frac{1}{2}(850 - 5x)$

Area $= A = xy$

$= \frac{x(850 - 5x)}{2}$

$= \frac{850}{2}x - \frac{5}{2}x^2$

$= 425x - \frac{5}{2}x^2$

$\Rightarrow \frac{dA}{dx} = -5x + 425 \overset{SET}{=} 0$

$-5x = -425$

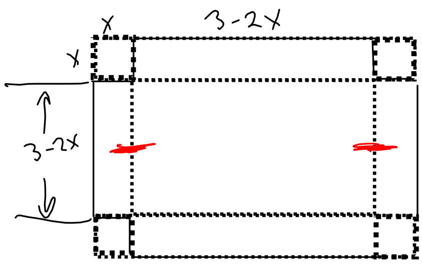
$x = \frac{-425}{-5} = \boxed{85 = x}$

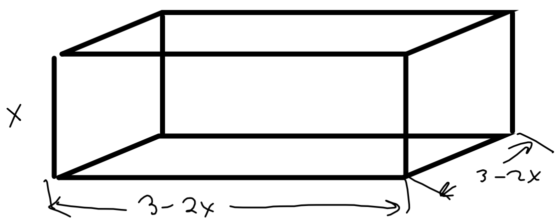
$\Rightarrow y = \frac{1}{2}(850 - 5(85))$

$= \frac{1}{2}(850 - 425) = \boxed{\frac{425}{2} = y}$

4. Question Details SCalc8 3.7.012. [3943251]

Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.





$$V = x(3-2x)^2 = x(2x-3)^2$$

$$\frac{dV}{dx} = 1(2x-3)^2 + x(2(2x-3))(2)$$

$$= 4x^2 - 12x + 9 + 4x(2x-3)$$

$$= 4x^2 - 12x + 9 + 8x^2 - 12x$$

$$= 12x^2 - 24x + 9 \overset{\text{SET}}{=} 0$$

$$a = 12, b = -24, c = 9$$

$$b^2 - 4ac = (-24)^2 - 4(12)(9)$$

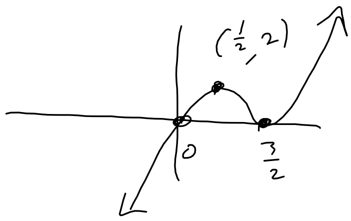
$$= 576 - 432$$

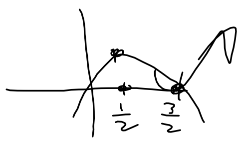
$$= 144$$

$$\begin{array}{r} 24 \\ \times 24 \\ \hline 96 \\ 48 \\ \hline 576 \end{array} \qquad \begin{array}{r} 48 \\ \times 9 \\ \hline 432 \end{array}$$

$$x = \frac{24 \pm \sqrt{144}}{2(12)} = \frac{24 \pm 12}{24} = \frac{2 \pm 1}{2} \to \frac{3}{2}, \frac{1}{2}$$

$$x = \frac{3}{2}, \quad x = \frac{1}{2} \qquad \boxed{x = \frac{1}{2}}$$

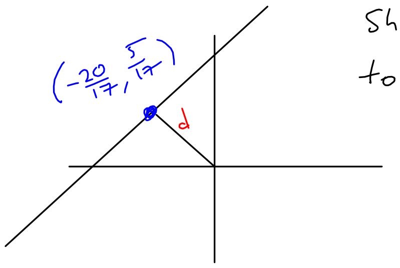




$$\frac{1}{2}(3-1)^2 = 2 = \text{Vol.}$$

5. Question Details SCalc8 3.7.021.

Find the point on the line $y = 4x + 5$ that is closest to the origin.



Shortest distance is on a line perpendicular to $y=4x+5$.

$y = -\frac{1}{4}(x-0)+0 = -\frac{1}{4}x$

where's it intersect $y=4x+5$?

$-\frac{1}{4}x = 4x+5$

$-x = 16x+20$

$-17x = 20$

$x = -\frac{20}{17} \Rightarrow y = 4\left(-\frac{20}{17}\right)+5 = \frac{-80+85}{17} = \frac{5}{17}$

How far is $y = 4x+5$ from $(0,0)$

$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$\sqrt{(x-0)^2+(4x+5-0)^2}$ to minimize,

minimize its square.

$D = d^2 = x^2+(4x+5)^2 = x^2+16x^2+40x+25$

$= 17x^2+40x+20$

$\frac{dD}{dx} = 34x+40 \overset{SET}{=} 0$

$\Rightarrow 34x = -40$

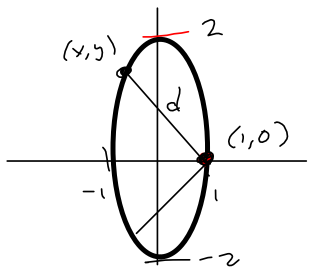
$x = -\frac{40}{34} = \boxed{-\frac{20}{17} = x}$

$\Rightarrow \boxed{y = \frac{5}{17}} \Rightarrow (x,y) = \left(-\frac{20}{17}, \frac{5}{17}\right)$

6. Question Details SCalc8 3.7.023.

Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad x^2+\frac{y^2}{4}=1$



distance $= d = \sqrt{(x-1)^2+(y-0)^2}$

$d^2 = D = (x-1)^2+y^2$

$4x^2+y^2 = 1$

$y^2 = 1-4x^2$

$y = \pm\sqrt{1-4x^2} \Rightarrow D = (x-1)^2+\left(\sqrt{1-4x^2}\right)^2$

Take top half.

$= x^2-2x+1+1-4x^2$

$= -3x^2-2x+2$

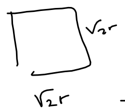
$\frac{dD}{dx} = -6x-2 \overset{SET}{=} 0 \Rightarrow x = -\frac{1}{3} \Rightarrow y = \sqrt{1-4\left(-\frac{1}{3}\right)^2} = \sqrt{1-4\left(\frac{1}{9}\right)} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$

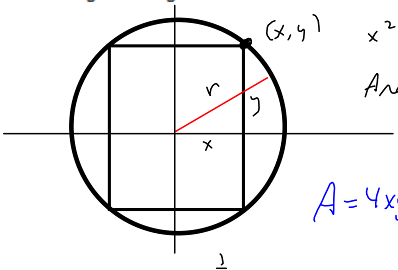
$$\boxed{\text{So, } (x,y) = \left(-\frac{1}{3}, \frac{\sqrt{5}}{3}\right) \text{ or } \left(-\frac{1}{3}, -\frac{\sqrt{5}}{3}\right) \text{ by symmetry.}}$$

7. Question Details — SCalc8 3.7.025.

Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

$2r^2$





$x^2+y^2=r^2 \Rightarrow y=\pm\sqrt{r^2-x^2}$

$=\sqrt{r^2-x^2}$ when above x-axis.

Area $=4xy$

$A=4xy=4x\sqrt{r^2-x^2}$
$=4x(r^2-x^2)^{\frac{1}{2}}$

$$\frac{dA}{dx}=4(r^2-x^2)^{\frac{1}{2}}+(4x)\left(\tfrac{1}{2}(r^2-x^2)^{-\frac{1}{2}}(-2x)\right)$$

$$=\frac{4(r^2-x^2)^{\frac{1}{2}}}{1}\cdot\frac{\sqrt{r^2-x^2}}{\sqrt{r^2-x^2}}-\frac{4x^2}{\sqrt{r^2-x^2}}$$

$$=\frac{4(r^2-x^2)-4x^2}{\sqrt{r^2-x^2}}=\frac{4r^2-4x^2-4x^2}{\sqrt{r^2-x^2}}$$

$$=\frac{4r^2-8x^2}{\sqrt{r^2-x^2}}\overset{\text{SET}}{=}0$$

$8x^2=4r^2$

$x^2=\frac{r^2}{2}$

$x=\pm\frac{r}{\sqrt{2}}=\pm\frac{\sqrt{2}\,r}{2}$

$y=\sqrt{r^2-x^2}$
$=\sqrt{r^2-\left(\frac{r}{\sqrt{2}}\right)^2}$
$=\sqrt{r^2-\frac{r^2}{2}}$
$=\sqrt{\frac{1}{2}r^2}=\frac{1}{\sqrt{2}}r$ if we take $r>0$.

$(x,y)=\left(\frac{1}{\sqrt{2}}r,\ \frac{1}{\sqrt{2}}r\right)$

$A=4xy=4\left(\frac{1}{\sqrt{2}}r\right)\left(\frac{1}{\sqrt{2}}r\right)$
$=\frac{4}{2}r^2=\boxed{2r^2=\text{Area}}$

8. Question Details — SCalc8 3.7.034.MI. [3353864]

A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle. See the figure below.) If the perimeter of the window is 24 ft, find the value of x so that the greatest possible amount of light is admitted.

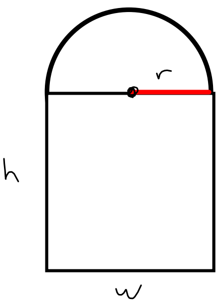
$x \to w$

Maximize Area

$\boxed{w=\frac{48}{\pi+4}}$

$\boxed{\text{So, } h=\frac{24}{\pi+4}.}$

See Maple Transcript.



$P=24=2h+w+\frac{1}{2}(2\pi r)$

$2h+w+\pi\left(\frac{w}{2}\right)=24$

$2h+\left(\frac{\pi}{2}+1\right)w=24$ = Perimeter

Eliminate a variable.

$\left(\frac{\pi}{2}+1\right)w=24-2h$

$\boxed{w=\frac{24-2h}{\frac{\pi}{2}+1}}$

Bad choice. (Legal, but hard.)

Area $=hw+\frac{1}{2}\pi r^2$
$=hw+\frac{1}{2}\pi\left(\frac{w}{2}\right)^2$
$=hw+\frac{1}{8}\pi w^2$

See Maple!

Perimeter $=P=2h+w+\frac{1}{2}\pi w=24$

$\Rightarrow h=12-\frac{1}{2}w-\frac{1}{4}\pi w$

$\Rightarrow A=\left(12-\frac{1}{2}w-\frac{1}{4}\pi w\right)w+\frac{1}{8}\pi w^2$

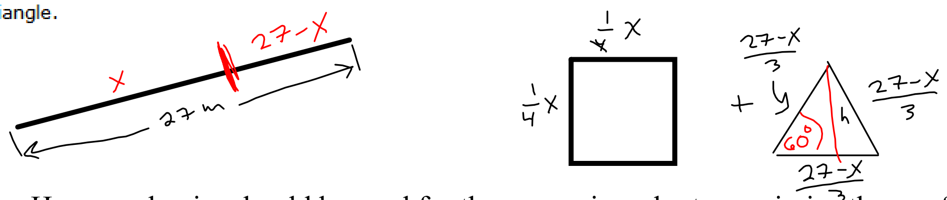
$=12w-\frac{1}{2}w^2-\frac{1}{8}\pi w^2$

$\Rightarrow \frac{dA}{dw}=12-w-\frac{1}{4}\pi w\overset{\text{SET}}{=}0$

$\Rightarrow w=\frac{48}{4+\pi}\Rightarrow h=\frac{24}{\pi+4}$

9. Question Details SCalc8 3.7.037. [3353592

A piece of wire 27 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle.



How much wire should be used for the square in order to maximize the area?

.....minimize the area?

Total area is $\left(\frac{1}{4}x\right)^2 + \frac{\sqrt{3}}{36}(x-27)^2 = A$

$A_2 + A_1 =$

$\frac{h}{\frac{27-x}{3}} = \sin 60^\circ = \frac{\sqrt{3}}{2}$

$h = \frac{27-x}{3}\cdot\frac{\sqrt{3}}{2} = \frac{27\sqrt{3}-\sqrt{3}x}{6}$

$A_1 = \frac{1}{2}bh = \frac{1}{2}\left(\frac{27-x}{3}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{27-x}{3}\right)$

$= \frac{\sqrt{3}}{36}(x-27)^2$ of triangle

$\Rightarrow \frac{dA}{dx} = 2\left(\frac{1}{4}x\right)^1\left(\frac{1}{4}\right) + \frac{\sqrt{3}}{36}\left(2(x-27)\right)$

$= \frac{1}{8}x + \frac{\sqrt{3}}{18}(x-27)$

$= \frac{1}{8}x + \frac{\sqrt{3}}{18}x - \frac{3\sqrt{3}}{2} \overset{SET}{=} 0$

2 | 18, 3 | 9, 3 2 | 8, 2 | 4, 2 72=LCD

$\frac{1}{8}\cdot\frac{9}{9} + \frac{\sqrt{3}}{18}\cdot\frac{4}{4} = \frac{9+4\sqrt{3}}{72}$

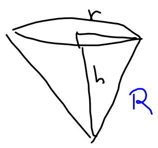
$\Rightarrow \left(\frac{1}{8}+\frac{\sqrt{3}}{18}\right)x = \frac{3\sqrt{3}}{2}$

$x = \left(\frac{3\sqrt{3}}{2}\right)\left(\frac{1}{\frac{1}{8}+\frac{\sqrt{3}}{18}}\right) = \frac{3\sqrt{3}}{2}\left(\frac{72}{\sqrt{3}+1}\right) = \boxed{\frac{36\sqrt{3}}{\sqrt{3}+1} = x}$

Should maximize the area.

10. Question Details SCalc8 3.7.041. [3354034]

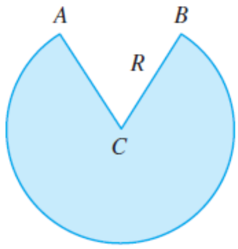
A cone-shaped drinking cup is made from a circular piece of paper of radius *R* by cutting out a sector and joining the edges *CA* and *CB*. Find the maximum capacity of such a cup.



$V = \frac{1}{3}\pi r^2 h$

$r^2 + h^2 = R^2$

$r^2 = R^2 - h^2$



$V = \frac{1}{3}\pi (R^2 - h^2)h$

$= \frac{1}{3}\pi \left[R^2 h - h^3\right] \Rightarrow$

$\frac{dV}{dh} = \frac{1}{3}\pi\left[R^2 - 3h^2\right] \overset{\text{SET}}{=} 0$

$\Rightarrow 3h^2 = R^2$

$\Rightarrow h^2 = \frac{R^2}{3}$

$\Rightarrow \boxed{h = \frac{R}{\sqrt{3}}} \Rightarrow r^2 = R^2 - h^2 = R^2 - \left(\frac{R}{\sqrt{3}}\right)^2$

$= R^2 - \frac{R^2}{3} = \boxed{\frac{2R^2}{3} = r^2}$

$r = \frac{\sqrt{2}R}{\sqrt{3}}$

$\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{\sqrt{2}}{\sqrt{3}}R\right)^2\left(\frac{R}{\sqrt{3}}\right)$

$= \frac{1}{3}\pi \frac{2R^2}{3} \cdot \frac{R}{\sqrt{3}} = \boxed{\frac{2}{9\sqrt{3}}\pi R^3}$