Section 3.4

Limits at infinity; Horizontal Asymptotes

$\boxed{1}$ **Intuitive Definition of a Limit at Infinity** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Intuitive Definition of a Limit at NEGATIVE Infinity

$\boxed{2}$ **Definition** Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

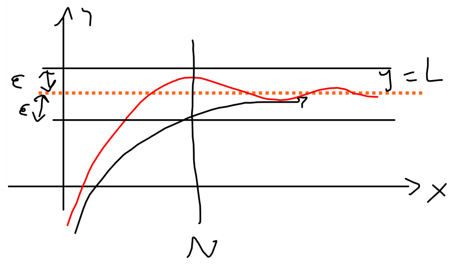
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

5 Precise Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$



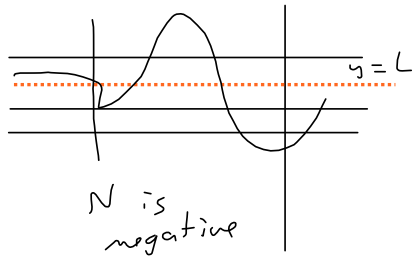
Precise Definition of a Limit at NEGATIVE Infinity:

6 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x < N \quad \text{then} \quad |f(x) - L| < \varepsilon$$



Claim: $\displaystyle\lim_{x\to\infty} \frac{1}{x^{\frac{2}{3}}} = 0$

Scratch

Want: $\left| \frac{1}{x^{2/3}} - 0 \right| = \frac{1}{x^{2/3}} < \varepsilon \implies$

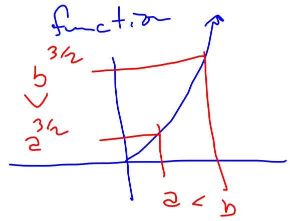
$1 < \varepsilon x^{2/3}$

$\implies \frac{1}{\varepsilon} < x^{2/3}$

$\left(\frac{1}{\varepsilon}\right)^{\frac{3}{2}} < \left(x^{2/3}\right)^{3/2}$

$N = \left(\frac{1}{\varepsilon}\right)^{\frac{3}{2}} < x$

NOTE $x^{3/2}$ is increasing function



Proof Let $\varepsilon > 0$ be given. Define $N = \left(\frac{1}{\varepsilon}\right)^{\frac{3}{2}}$. Then if $x > N$, we have

$\left| \frac{1}{x^{2/3}} - 0 \right| = \frac{1}{x^{2/3}} < \frac{1}{N^{2/3}} = \frac{1}{\left(\left(\frac{1}{\varepsilon}\right)^{3/2}\right)^{2/3}} = \frac{1}{\left(\frac{1}{\varepsilon}\right)} = 1 \cdot \frac{\varepsilon}{1} = \varepsilon.$

$\left(x > N \implies \frac{1}{x} < \frac{1}{N} \right)$

$100 > 10 \to \frac{1}{100} < \frac{1}{10}$

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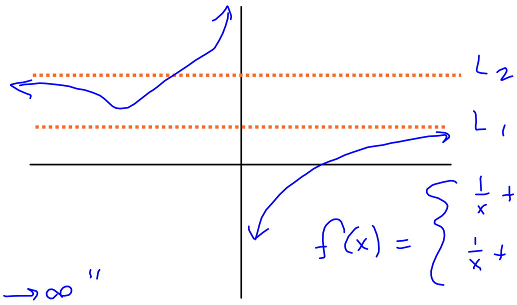
3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L_1 \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L_2$$



$$f(x) = \begin{cases} \frac{1}{x} + 5 & \text{if } x \geq 0 \\ \frac{1}{x} + 8 & \text{if } x < 0 \end{cases}$$

Real

4 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

" $x \to \infty$ " means "eventually"

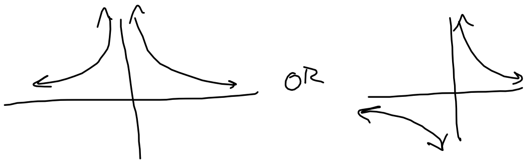
Proved this result for $r = 2/3$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Need this for $x < 0$

$\frac{1}{x^{\sqrt{2}}}$ is tough to define for $x < 0$.



7 Definition of an Infinite Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

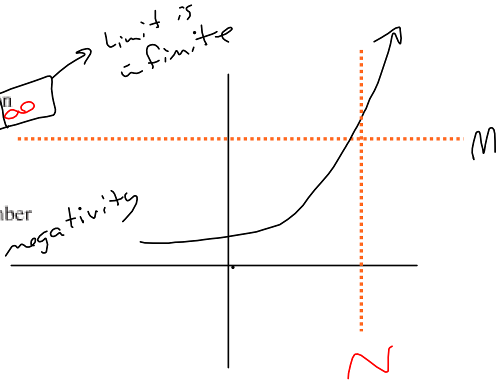
$\lim_{x \to \boxed{-\infty}} f(x) = \boxed{\infty}$

Limit is $\neq$ finite

at negativity

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \quad \text{then} \quad f(x) > M$$



1. Question Details SCalc8 3.4.002. [3354015]

(a) Can the graph of $y = f(x)$ intersect a vertical asymptote?

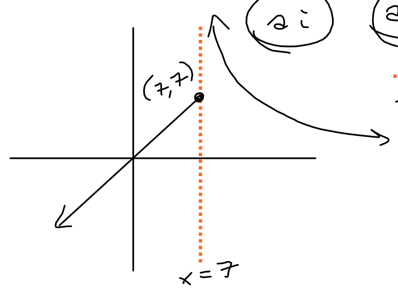
LOL! Yes, but I had to go piecewise. $f(x) = \begin{cases} x & \text{if } x \le 7 \\ \frac{1}{x-7} & \text{if } x > 7 \end{cases}$

Can it intersect a horizontal asymptote? Generally No.

(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? (Select all that apply.)

Up to 2.

2 i



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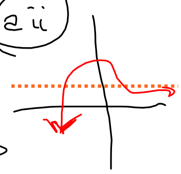
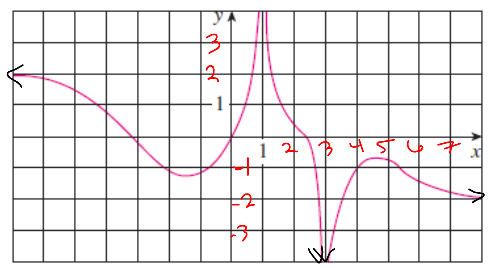


Question Details SCalc8 3.4.003. [3353795

For the function f whose graph is given, state the following.



(a) $\lim_{x \to \infty} f(x) = -2$

(b) $\lim_{x \to -\infty} f(x) = 2$

(c) $\lim_{x \to 1} f(x) = \infty$ *

(d) $\lim_{x \to 3} f(x) = -\infty$

(e) the equations of the asymptotes $\boxed{\text{H.A. } y = 2,\ y = -2}$ $\boxed{\text{V.A. } x = 1,\ x = 3}$

* often when $\lim_{x \to *} f(x) = \infty$, we say "Limit $\nexists$"

3. Question Details SCalc8 3.4.005. [3353740]

Guess the value of the limit

$$\lim_{x\to\infty} \frac{x^5}{5^x}$$

by evaluating the function $f(x) = x^5/5^x$ for x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50, and 100. Use a graph of f to support your guess.

$f(10) > f(100) > f(1000) > 0$

stuff $\times 10^{-684} \approx 0$!

$evalf([f(5), f(10), f(100), f(1000)])$

$[1., 0.01024000000, 1.267650600\ 10^{-60}, 1.071508607\ 10^{-684}]$

The Perfect time to talk about compound interest is before #4:

A = future value of an account.
P = Principal
r = annual rate of interest
t = time, in years
m = # of periods per year.

$A = P\left(1+\frac{r}{m}\right)^{mt} \xrightarrow{m\to\infty} Pe^{rt}$

$m = 365$ is very close to Pe^{rt}

$\left(1+\frac{r}{m}\right)^{mt} = \left(1+\frac{r}{m}\right)^{\frac{m}{r}\cdot rt}$

WTS: $\left(1+\frac{r}{m}\right)^{\frac{m}{r}} \xrightarrow{m\to\infty} e$

"Proof" by symbolic algebra on a 'puter. (See Video!)

4. Question Details SCalc8 3.4.006. [3353949]

Consider the function below.

$f(x) = \left(1 - \frac{4}{x}\right)^x = \left(1+\left(-\frac{4}{x}\right)\right)^{\left(-\frac{x}{4}\right)(-4)} = \left(\left(1+\left(-\frac{4}{x}\right)\right)^{\left(-\frac{x}{4}\right)}\right)^{-4} \xrightarrow{x\to\infty} e^{-4}$

(a) Use a graph to estimate the value of the limit of $\lim_{x\to\infty} f(x)$ correct to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

$$\lim_{m\to\infty} \left(1+\frac{r}{m}\right)^{\frac{m}{r}} = e$$

Plug in as many as you needed to power up your intuition.

$e^{-4} \approx 0.01831563889$

TABLE of ONE:

$f(1000) \approx 0.01830098833$

5. Question Details SCalc8 3.4.007. [3353653]

Evaluate the limit using the appropriate properties of limits.

$$\lim_{x\to\infty} \frac{2x^2 - 5}{5x^2 + x - 3} = \frac{2}{5}$$

$$\frac{\text{degree num} = n}{\text{degree den} = n}$$

Look at degree-n terms!

$$\frac{2x^2 - 5}{5x^2 + x - 3} = \frac{\cancel{x^2}\left(2 - \frac{5}{x^2}\right)}{\cancel{x^2}\left(5 + \frac{1}{x} - \frac{3}{x^2}\right)} \xrightarrow{x\to\infty} \frac{2}{5}$$

$\frac{5}{x^2} \to 0$, $\frac{1}{x} \to 0$, $\frac{3}{x^2} \to 0$

$$r > 0 \Rightarrow \frac{1}{x^r} \xrightarrow{x\to\infty} 0$$

6. SCalc8 3.4.011. [3354001]

Find the limit, if it exists. (If an answer does not exist, enter DNE.)

$$\lim_{x \to -\infty} \frac{x-1}{x^2+6} = 0$$

$x^2 > x$, eventually.

Recall degree denom > degree numer (like #6), then $\lim_{|x| \to \infty} f(x) = 0$. and $f(x)$ is a PROPER rational function

7. Question Details SCalc8 3.4.013. [3353759]

Find the limit, if it exists.

$$\lim_{t \to \infty} \frac{\sqrt{t} + t^2}{4t - t^2} \quad \cdot \frac{\deg = 2}{\deg = 2}$$

Look at the highest-degree terms

$$\frac{t^2}{-t^2} = \boxed{-1}$$

8. Question Details SCalc8 3.4.016. [3353898]

Find the limit, if it exists.

$$\lim_{x \to \infty} \frac{x^3}{\sqrt{x^6+1}}$$

$$\frac{x^3}{\sqrt{x^6+1}} = \frac{x^3}{\sqrt{x^6\left(1+\frac{1}{x^6}\right)}} = \frac{x^3}{\sqrt{x^6}\sqrt{1+\frac{1}{x^6}}}$$

$$= \frac{x^3}{|x^3|\sqrt{1+\frac{1}{x^6}}} \qquad \sqrt{x^6} = (x^6)^{\frac{1}{2}} = x^{6\cdot\frac{1}{2}} = x^3$$

$$= \frac{x^3}{x^3\sqrt{1+\frac{1}{x^6}}} = \frac{1}{\sqrt{1+\frac{1}{x^6}}} \xrightarrow{x \to \infty} \frac{1}{\sqrt{1}} = \boxed{1}$$

because $x > 0$ so $|x| = x$ & $|x^3| = |x|^3$

$x < 0 \implies |x| = -x$

9. Question Details SCalc8 3.4.021. [3353893]

Find the limit, if it exists.

$\lim_{x \to \infty} \left(\sqrt{4x^2 + x} - 2x \right)$ $= \infty - \infty$ is indeterminate form.

old trick: Conjugate idea:

$$\left(\frac{\sqrt{4x^2+x} - 2x}{1}\right)\left(\frac{\sqrt{4x^2+x} + 2x}{\sqrt{4x^2+x} + 2x}\right) = \frac{4x^2 + x - 4x^2}{\sqrt{4x^2+x} + 2x}$$

$$= \frac{x}{\sqrt{4x^2+x} + 2x} = \frac{x}{|x|\sqrt{4+\frac{1}{x}} + 2x} = \frac{\cancel{x}}{\cancel{x}\left(\sqrt{4+\frac{1}{x}} + 2\right)}$$

$\frac{\infty}{\infty + \infty}$?!

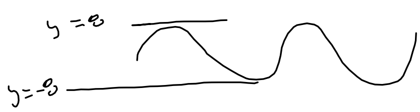
$x > 0 \Rightarrow |x| = x$

$$\frac{1}{\sqrt{4+\frac{1}{x}} + 2} \xrightarrow{x \to \infty} \frac{1}{\sqrt{4+0} + 2} = \boxed{\frac{1}{4}}$$

10. Question Details SCalc8 3.4.024. [3353786]

Find the limit, if it exists.

$$\lim_{x \to \infty} 8 \cos x \quad \nexists$$



Never settles down to one y-value.

11. Question Details SCalc8 3.4.026.MI.

Find the limit, if it exists.

$$\lim_{x \to \infty} \sqrt{x^2 + 5} = \infty$$

12. Question Details SCalc8 3.4.027.

Find the limit, if it exists.

$$\lim_{x \to -\infty} (x^2 + 2x^7) = -\infty$$

Book : $x^2 + 2x^7 = x^2\left(1 + 2x^5\right)$

$$(\infty)(-\infty) \text{ " = " } -\infty$$

Me : $|2x^7| >>>> |x^2|$ for "Big" x.

$2x^7 \xrightarrow{x \to -\infty} -\infty$ much "bigger infinity." than x^2

13. Question Details SCalc8 3.4.034. [3353745]

(a) Use a graph of

$$f(x) = \sqrt{2x^2 + 9x + 2} - \sqrt{2x^2 + 2x + 1} = \sqrt{A} - \sqrt{B} = \left(\frac{\sqrt{A} - \sqrt{B}}{1}\right)\left(\frac{\sqrt{A} + \sqrt{B}}{\sqrt{A} + \sqrt{B}}\right),$$

to estimate the value of $\lim_{x \to \infty} f(x)$ to one decimal place.

etc., if by hand.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Find the exact value of the limit.

Desmos Grapher.

a 2.5 ish

b 2.475 ish

c $\frac{7\sqrt{2}}{4}$

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14. Question Details

Find the horizontal and vertical asymptotes of the curve.

$$y = \frac{2x^2 + x - 5}{x^2 + x - 2}$$

H.A. $\frac{2x^2}{x^2} = \boxed{2 = y}$

V.A. Find zeros of denom.

$x^2+x-2 = (x+2)(x-1) \overset{SET}{=} 0 \implies \boxed{x=-2,\ x=1}$ is good.

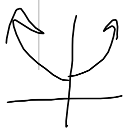
$2x^2+x-5 = (2x\quad)(x\quad)\ldots$

$b^2-4ac = 1^2-4(2)(-5) = 1+40 = 41$

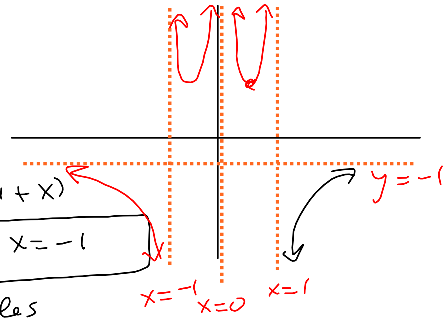
$x = \frac{-1 \pm \sqrt{41}}{2(2)}$ No shared zeros so

15. Question Details SCalc8 3.4.038. [3353930]

Find the horizontal and vertical asymptotes of the curve.



$$y = \frac{2 + x^4}{x^2 - x^4}$$



$x^2 - x^4 = x^2(1-x^2) = x^2(1-x)(1+x)$

V.A. $\boxed{x=0,\ x=1,\ x=-1}$

No holes

H.A. : $\boxed{y=-1}$

16. Question Details SCalc8 3.4.039. [3353658]

Find the horizontal and vertical asymptotes of the curve.

$$y = \frac{x^2 - x}{x^2 - 4x + 3} = \frac{x(x-1)}{(x-3)(x-1)}$$

Has a hole!

$\mathcal{D} = \mathbb{R}\setminus\{1,3\}$ Instinct says V.A.: $\boxed{\cancel{x=1},\ x=3}$

But! x-1 is a factor, upstairs.

It cancels w/ x-1, downstairs

$\frac{x\cancel{(x-1)}}{(x-3)\cancel{(x-1)}} = \frac{x}{x-3},\ x \neq 1$

looks like this, but a hole @ $x=1$:

Plug in $x=1$: $\frac{1}{1-3} = \frac{1}{-2} = -\frac{1}{2}$

$\rightsquigarrow (1, -\frac{1}{2}) = $ Hole.

$\boxed{\text{H.A.: } y=1}$ (from $\frac{x^2}{x^2}=1$)

17. Question Details SCalc8 3.4.043. [3353810]

Let P and Q be polynomials with positive coefficients. Consider the limit below.

$$\lim_{x \to \infty} \frac{P(x)}{Q(x)}$$

(a) $\lim_{x\to\infty} \frac{P(x)}{Q(x)} = 0$, $y=0$ is H.A.

(a) Find the limit if the degree of P is less than the degree of Q. $n < m \longrightarrow$ Proper $\boxed{y=0}$ is H.A.

$n > m$ (b) Find the limit if the degree of P is greater than the degree of Q.

$$\frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \frac{\sum_{k=0}^{n} a_k x^k}{\sum_{k=0}^{m} b_k x^k} = \frac{\sum_{k=0}^{n} a_{n-k} x^{n-k}}{\sum_{k=0}^{m} b_{m-k} x^{m-k}}$$

Ascending order Descending order.

(b) $\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \infty$

18. Question Details SCalc8 3.4.045. [4004969]

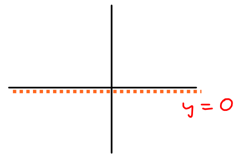
Find a formula for a function f that satisfies the following conditions.

$$\lim_{x \to \pm\infty} f(x) = 0, \quad \lim_{x \to 0} f(x) = -\infty, \quad f(4) = 0$$

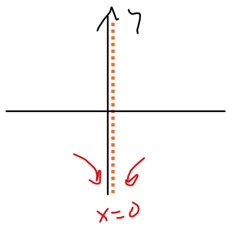
$$\lim_{x \to 5^-} f(x) = \infty, \quad \lim_{x \to 5^+} f(x) = -\infty$$

$\lim_{x \to \pm\infty} f(x) = 0$ Proper $\frac{a_n x^n + \cdots}{b_m x^m + \cdots}$, $m > n$

$\lim_{|x| \to \infty}$



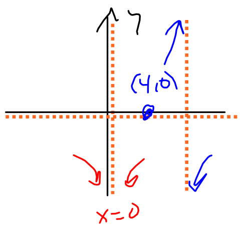
$\lim_{x \to 0} f(x) = -\infty$



$\lim_{x \to 5^-} f(x) = \infty$

$\lim_{x \to 5^+} f(x) = -\infty$

$f(4) = 0$



$$\boxed{\frac{(x-4)}{x^2(x-5)} = f(x)}$$

19. Question Details SCalc8 3.4.046. [3354050]

Find a formula for a function that has vertical asymptotes $x = 5$ and $x = 9$ and horizontal asymptote $y = 5$.

$$\boxed{\frac{5x^2}{(x-5)(x-9)}} = \frac{5x^2}{x^2+\cdots} \xrightarrow{x \to \infty} 5\frac{x^2}{x^2} = 5 = y$$

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20. Question Details SCalc8 3.4.052. [3394377]

Find the limits as $x \to \infty$ and as $x \to -\infty$.

$y = f(x) = 3x^3 - x^4$ Eventually dominated by $-x^4$

Its end behavior is

$-\infty$ $-\infty$

$\lim_{|x| \to \infty} f(x) = -\infty.$

21. Question Details SCalc8 3.4.055. [3394382]

Find the limits as $x \to \infty$ and as $x \to -\infty$.

$y = f(x) = (3 - x)(1 + x)^2(1 - x)^4$ "=" $(-x)(x)^2(-x)^4 = -x^7$

$= -x^7 + \text{lower degree stuff.}$

x^7 $-x^7$

$f(x) \xrightarrow{x \to \infty} -\infty$

$f(x) \xrightarrow{x \to -\infty} +\infty$

22. Question Details SCalc8 3.4.064. [3353885]

A tank contains 9000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. The concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{360 + t} \xrightarrow{t \to \infty} 30$$

As $t \to \infty$, what does the concentration approach?

23. Question Details SCalc8 3.

A graphing calculator is recommended.

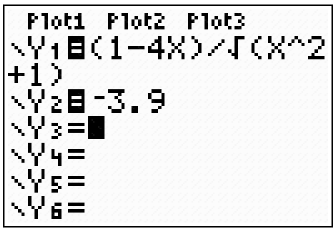
Desmos won't graph $y = -3.9, -4.1$, etc.

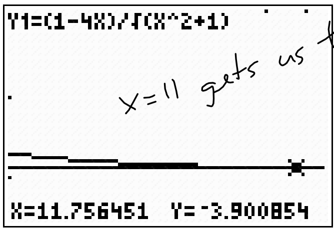
For the limit

$$\lim_{x\to\infty} \frac{1-4x}{\sqrt{x^2+1}} = -4$$

illustrate the definition by finding the smallest integer values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.05$.

want $f(x) < -3.95$





$x = 11$ gets us there

$x = 22$ gets us below -3.95.

24. Question Details

How large do we have to take x so that $1/\sqrt{x} < 0.1$?

$$\frac{1}{\sqrt{x}} < .1 = \frac{1}{10}$$

$$1 < \frac{1}{10}\sqrt{x}$$

$$\frac{1}{10}\sqrt{x} > 1$$

$$\sqrt{x} > 10$$

$$(\sqrt{x})^2 > 10^2 = 100$$

$\boxed{x > 100 \text{ does it}}$