§3.4 Limits at Infinity

- Some College Algebra Review,
  Polynomials
  Sign Patterns
  Proper/Improper Rational Functions & their asymptotes.

Videos on College Algebra

- Throw some Calculus on top of that.

Some quick graph-sketching concepts (Review of Algebra), and sign-pattern demo.

$$x^2 - 6x + 7$$

$$f(x) = x^5 - 18x^4 + 124x^3 - 404x^2 + 615x - 350$$

$$= (x-2)(x-5)^2(x-3+\sqrt{2})(x-3-\sqrt{2})$$

Rational Zeros

$a_n x^n + \cdots + a_0$ Any $\frac{p}{q} \ni f\left(\frac{p}{q}\right) = 0$

will satisfy p is factor of a_0

q " " " a_n

$2 \,|\, 350$, $5 \,|\, 175$, $5 \,|\, 35$, 7

$a_n = a_5 = 1$

$a_0 = -350$

$\frac{p}{q}: \pm 2, \pm 5, \pm 7, \pm 10, \ldots$

Break it down by synthetic division

Divide by $x-2$ $\quad (x-2)(x^4 - 16x^3 + 92x^2 - 220x + 175)$

2	1	-18	124	-404	615	-350
		2	-32	184	-440	350
5	1	-16	92	-220	175	$0 = f(2)$
		5	-55	185	-175	
5	1	-11	37	-35	0	
		5	-30	35		
	1	-6	7	0		

Now we have $(x-2)(x-5)^2(x^2-6x+7)$

$x^2 - 6x + 7$

$b^2 - 4ac = (-6)^2 - 4(1)(7) = 36 - 28 = 8 \rightsquigarrow \sqrt{8} = 2\sqrt{2}$

$$x = \frac{6 \pm 2\sqrt{2}}{2(1)} = \frac{2(3 \pm \sqrt{2})}{2} = 3 \pm \sqrt{2}$$

$\Rightarrow x^2 - 6x + 7 = (x - (3+\sqrt{2}))(x - (3-\sqrt{2}))$

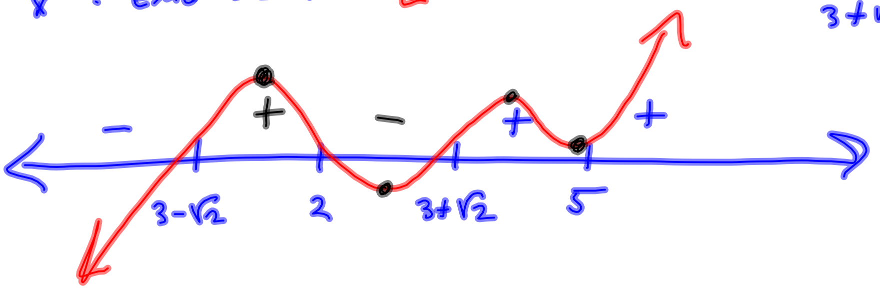
$(x-2)(x-5)^2(x-3-\sqrt{2})(x-3+\sqrt{2})$

$\sqrt{2} \approx 1.4$

$3 - \sqrt{2} \approx 1.6$

$3 + \sqrt{2} \approx 4.4$

x^5 : End-Behavior



End Behavior $\lim\limits_{x\to\pm\infty} f(x)$

$\lim\limits_{|x|\to\infty} f(x)$

Proper/Improper Rational Function.

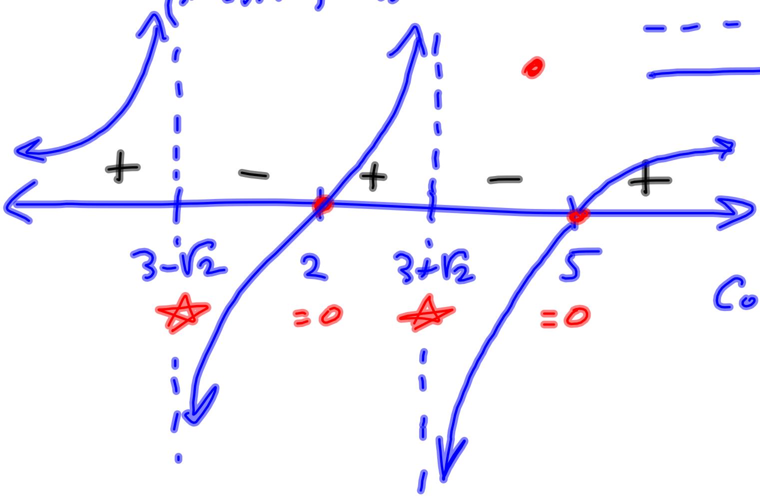
$$\lim_{x\to\infty} \frac{x^2-7x+10}{x^2+6x+7}$$

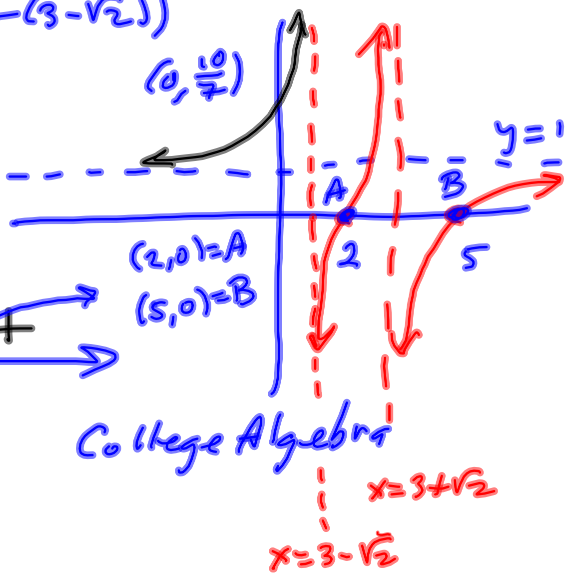
$$\frac{x^2}{x^2}=1 \xrightarrow{x\to\infty} \boxed{1=y} \text{ is H.A.}$$

$$x^2-7x-10 = (x-2)(x-5)$$

$$x^2+6x+7 = (x-(3+\sqrt{2}))(x-(3-\sqrt{2}))$$

$$y = \frac{(x-2)(x-5)}{(x-(3+\sqrt{2}))(x-(3-\sqrt{2}))}$$





$$f := x \rightarrow \frac{x}{x^2+1}$$

#47 from text

$$fp := \mathrm{D}(f)$$

$$x \rightarrow \frac{1}{x^2+1} - \frac{2x^2}{\left(x^2+1\right)^2}$$

$$normal\left(\frac{1}{x^2+1} - \frac{2x^2}{\left(x^2+1\right)^2}\right)$$

$$-\frac{x^2-1}{\left(x^2+1\right)^2}$$

$$fpp := \mathrm{D}(fp)$$

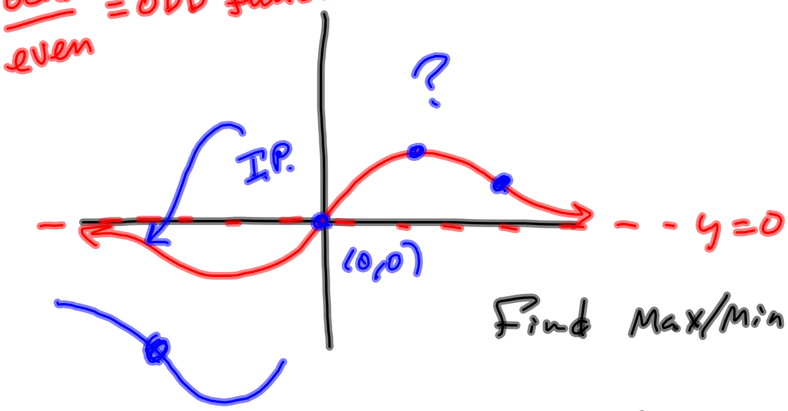
$$x \rightarrow -\frac{6x}{\left(x^2+1\right)^2} + \frac{8x^3}{\left(x^2+1\right)^3}$$

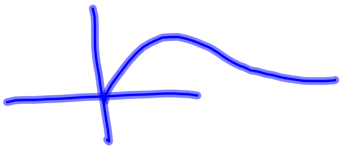
$$normal\left(-\frac{6x}{\left(x^2+1\right)^2} + \frac{8x^3}{\left(x^2+1\right)^3}\right)$$

$$\frac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3}$$

$f(x) = \frac{x}{x^2+1}$ Proper $y=0$ is H.A.

$\frac{odd}{even}$ = ODD func.





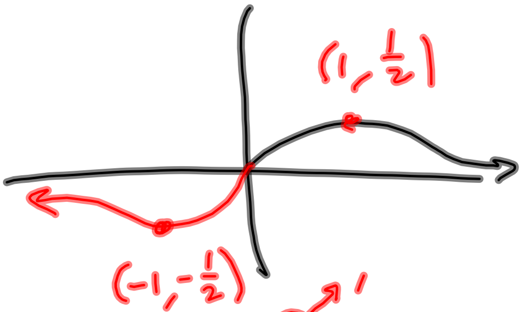
$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Find Max/Min, IP's.

$$f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

$-x^2+1 \overset{SET}{=} 0 \Rightarrow x^2 - 1 = 0 \Rightarrow (x-1)(x+1) = 0$

$x = \pm 1$ $f(1) = \frac{1}{1^2+1} = \frac{1}{2}$



$$f''(x) = \frac{-2x(x^2+1)^{2 \to 1} - (-x^2+1)(2\cancel{(x^2+1)}(2x))}{(x^2+1)^{4 \to 3}}$$

SWIDT?

$(-4x^3+4x)$

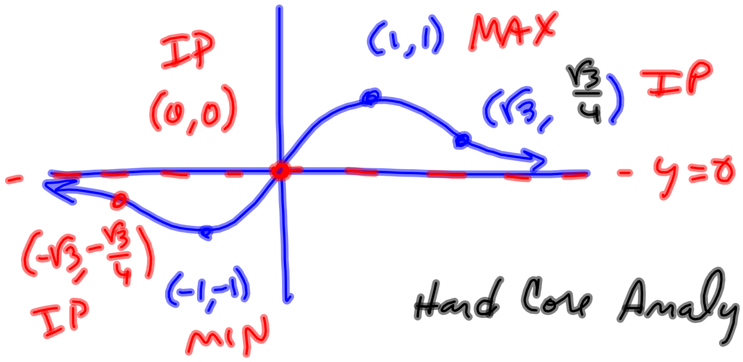
$$= \frac{-2x(x^2+1) - (-x^2+1)(4x)}{(\ )^3}$$

$$= \frac{-2x^3 - 2x + 4x^3 - 4x}{(\ )^3} = \frac{2x^3 - 6x}{(x^2+1)^3}$$

$\overset{SET}{=} 0 \Rightarrow 2x^3 - 6x = 0$

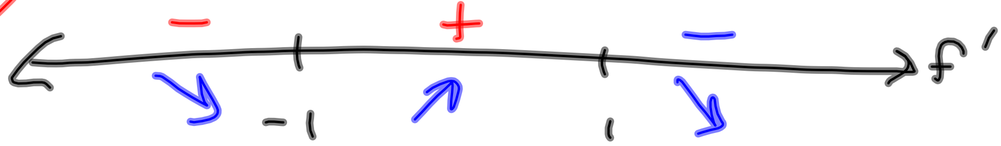
$2x(x^2-3) = 0$ $f(\sqrt{3}) = \frac{\sqrt{3}}{3+1} =$

$x=0$ OR $x = \pm\sqrt{3}$

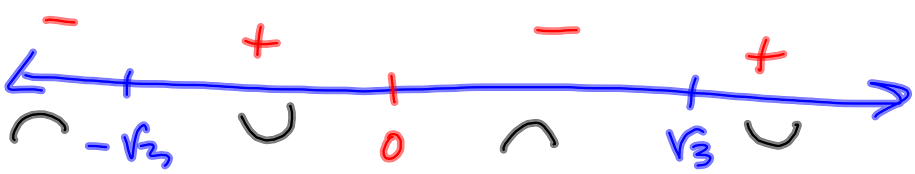


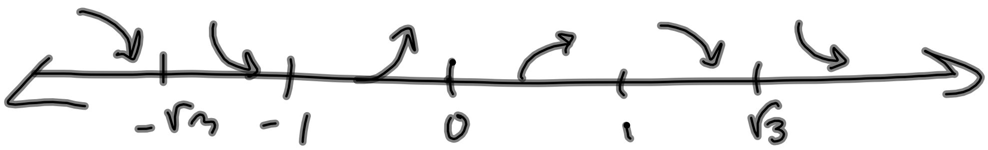
Hard Core Analysis.

$$\frac{-x^2+1}{(x^2+1)^2}$$



$$\frac{2x^3-6x}{(x^2+1)^3}$$





$$2x(x^2-3) = 2x(x-\sqrt{3})(x+\sqrt{3})$$

$$\frac{-x^2+1}{(x^2+1)^2} = -\frac{x^2-1}{(x^2+1)^2} = -\frac{(x-1)(x+1)}{(x^2+1)^2}$$

$$\lim_{x\to\infty} \frac{3x+5}{4-7x} = \lim_{x\to\infty} \frac{3x}{-7x} = -\frac{3}{7}$$

$$\lim_{x\to\infty} \frac{9x+4}{x^2+1} = 0$$

(This is a proper rational function.)

H.A.: $y=0$

$$\lim_{x\to\infty} \frac{(2x+3)(3x-7)^2}{10x^3+9x^2-3x+4}$$

$$= \lim_{x\to\infty} \frac{18x^3+\cdots}{10x^3+\cdots} = \frac{18}{10} = \boxed{\frac{9}{5}}$$

$$\sqrt{16x^2-2x} - 4x = \boxed{\text{Want } \lim_{x\to\infty}}$$

$$\frac{2x}{16x^2} = \frac{1}{8x}$$

$$\frac{\left(\sqrt{16x^2-2x} - 4x\right)}{1} \cdot \frac{\sqrt{16x^2-2x} + 4x}{\sqrt{16x^2-2x} + 4x}$$

$$\sqrt{16x^2(\text{stuff})} = 4|x|\sqrt{\text{stuff}}$$

$$= \frac{16x^2 - 2x - 16x^2}{\sqrt{16x^2-2x} + 4x} = \frac{-2x}{\sqrt{16x^2-2x} + 4x} = \frac{-2x}{\sqrt{16x^2\left(1 - \frac{1}{8x}\right)} + 4x}$$

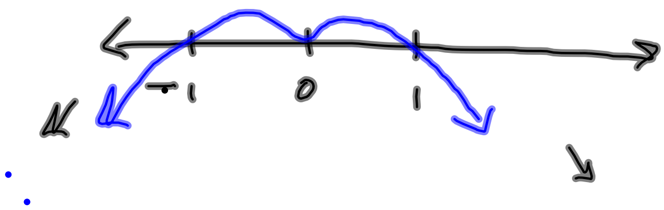
$$= \frac{-2x}{4|x|\sqrt{1 - \frac{1}{8x}} + 4x} = \frac{-2x}{4x\sqrt{1 - \frac{1}{8x}} + 4x} = \frac{-2x}{4x\left(\sqrt{1 - \frac{1}{8x}} + 1\right)}$$

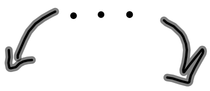
$$= \frac{-1}{2\left(\sqrt{1 - \frac{1}{8x}} + 1\right)} \xrightarrow{x\to\infty} \frac{-1}{2\left(\sqrt{1 - 0} + 1\right)} = -\frac{1}{2(2)} = \boxed{-\frac{1}{4}}$$

$\frac{1}{8x} \to 0$

#49? $y = x^4 - x^6 = -x^4(x^2-1) = -x^4(x-1)(x+1)$

$-x^6$





$f'(x) = 4x^3 - 6x^5 = -2x^3(3x^2-2) = -2x^3(\sqrt{3}\,x - \sqrt{2})(\sqrt{3}\,x + \sqrt{2})$

$x = 0, \pm\frac{\sqrt{6}}{3}$

$\sqrt{3}\,x - \sqrt{2} = 0$

$x = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$

$f''(x) = 12x^2 - 30x^4 = -6x^2(5x^2-2) \overset{SET}{=} 0$

$x = 0$

$5x^2 - 2 = 0$

$5x^2 = 2$

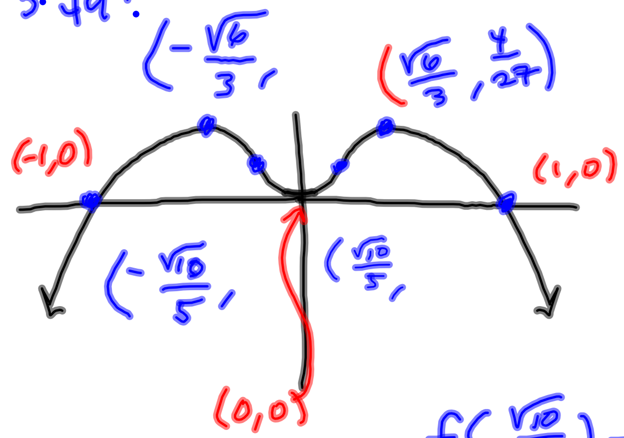
$\sqrt{x^2} = \sqrt{\frac{2}{5}}$ $\quad x^2 = \frac{2}{5}$

$|x| = \sqrt{\frac{2}{5}}$

$x = \pm\sqrt{\frac{2}{5}} = \pm\frac{\sqrt{10}}{5}$

3.4 #49?

$f\left(\frac{\sqrt{6}}{3}\right)$



$x^4 - x^6$

$f\left(\frac{\sqrt{10}}{5}\right) =$

$$f\left(\frac{\sqrt{6}}{3}\right) = \left(\frac{\sqrt{6}}{3}\right)^4 - \left(\frac{\sqrt{6}}{3}\right)^6$$

$$\frac{36}{3^4} - \frac{6^3}{3^6}$$

$$= \frac{2^2 \cdot 3^2}{3^4} - \frac{2^3 \cdot 3^3}{3^6}$$

$$= \frac{2^2}{3^2} - \frac{2^3}{3^3} = \frac{2^2 \cdot 3 - 2^3}{3^3}$$

$$= \frac{12-8}{27} = \frac{4}{27}$$

$2 \,\underline{)\, 36}$
$2 \,\underline{)\, 18}$
$3 \,\underline{)\, 9}$
3