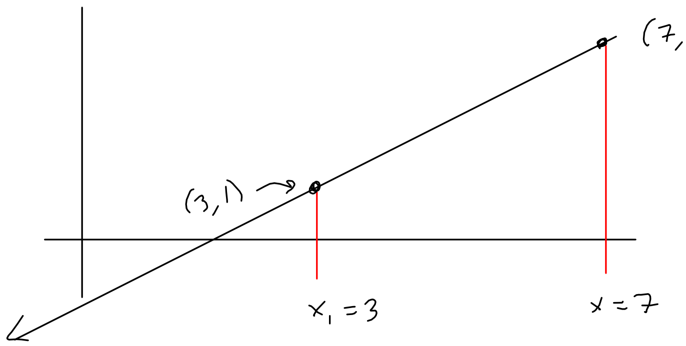
Section 2.9 Linear Approximations and Differentials



$y = 2x - 5$

How do you get from (x_1, y_1) to the next (x, y)-pair?

what's the new y-value?

$y = y_1 + m(x - x_1)$

$(x_1, y_1) = (3, 1)$

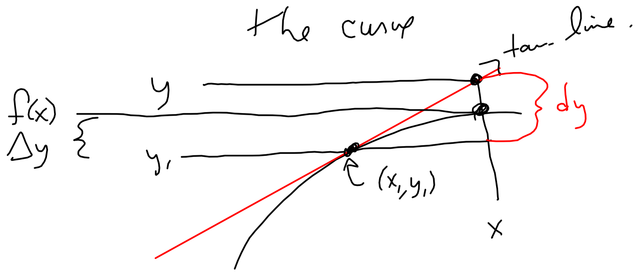
$y = 1 + 2(7-3)$

$= 1 + 2(4) = 9$

$2(7) - 5 = 9$

New y-value is old y-value + (slope)(horiz. dist.)

$y = m(x - x_1) + y_1$ We want to apply this to curves, approximating them by the tangent line. Over short distances, the tangent is very close to the curve



$y = f'(x_1)(x - x_1) + y_1$

$dy = f'(x)(x - x_1)$

$\Delta y = f(x) - f(x_1) \approx dy$

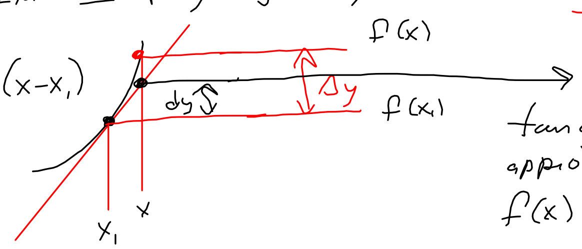
we call $x - x_1 = \boxed{\Delta x = dx}$

Identical!

In this pic, $dy > \Delta y$

$\Delta y \approx dy = f'(x_1)dx = m_{tan}(x - x_1)$

$\Delta y = f(x) - f(x_1)$



$L_{x_1}(x) =$ tangent line approx. to $f(x)$ @ $x = x_1$

1. SCalc8 2.9.002. (3354217)

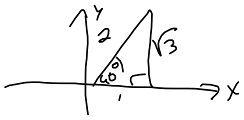
Find the linearization $L(x)$ of the function at a. $\rightarrow L_a(x)$

$f(x) = \sin(x), \quad a = \frac{\pi}{3}$

Linearization at a is the Tangent Line at a. That's all we're doing this entire section, once you clear away the trash.

$\boxed{y = f'(x_1)(x - x_1) + f(x_1)} = L_{x_1}(x)$

$\Rightarrow f'(x) = \cos(x) \Rightarrow f'(\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = \frac{1}{2} = m_{tan} \Rightarrow$ $y_1 = \sin(\frac{\pi}{3}) = f(\pi/3)$



$y = m_{tan}(x - x_1) + y_1$

$= \frac{1}{2}(x - \frac{\pi}{3}) + \frac{\sqrt{3}}{2}$ STOP!

$= \frac{1}{2}x - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$ Book Sux

2. SCalc8 2.9.003. (3354140)

Find the linearization $L(x)$ of the function at a.

$f(x) = \sqrt{x}, \quad a = 4$

$y = m(x - x_1) + y_1$

$L_{x_1}(x) = f'(x_1)(x - x_1) + f(x_1)$

$L_a(x) = f'(a)(x - a) + f(a)$

$f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

$\Rightarrow f'(4) = \frac{1}{2}(4)^{-\frac{1}{2}} = \frac{1}{2}\left(\frac{1}{\sqrt{4}}\right) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} = f'(a)$

$L_a(x) = f'(a)(x - a) + f(a)$

$\boxed{L_4(x) = \frac{1}{4}(x - 4) + 2}$

$f(a) = f(4) = \sqrt{4} = 2$

3. SCalc8 2.9.005. (3354568)

Find the linear approximation of the function $f(x) = \sqrt{1 - x}$ at $a = 0$.

Use $L(x)$ to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$. (Round your answers to four decimal places.)

Illustrate by graphing f and the tangent line.

why get cute w/ $\sqrt{1-x}$ @ $x=0$, when $\sqrt{x}$ @ $x=1$ is perfect for $\sqrt{.9}$ & $\sqrt{.99}$. Oh well.

$f(x) = (1-x)^{\frac{1}{2}} \Rightarrow$

$f'(x) = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) = \frac{-1}{2\sqrt{1-x}}$

$f'(a) = \frac{-1}{2\sqrt{1-0}} = f'(0) = -\frac{1}{2}$

$f(a) = \sqrt{1-0} = 1 = y_1$

$L_0(x) = -\frac{1}{2}(x - 0) + 1$

$= -\frac{1}{2}x + 1$

Got my x wrong due to unnecessary cuteness!

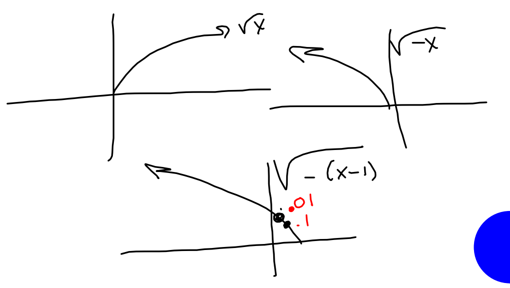
$\sqrt{.9} = \sqrt{1 - .1}$

$x = .1$!

$\sqrt{.99} = \sqrt{1 - .01}$

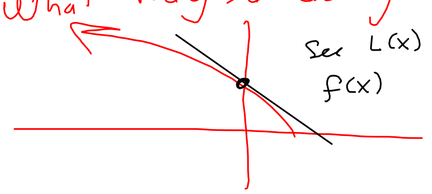
$L_a(.9) = (-\frac{1}{2})(\frac{1}{10}) + 1 = \frac{-1 + 20}{20} = \boxed{\frac{19}{20} = L(.1)}$

$L_0(.99) = (-\frac{1}{2})(\frac{1}{100}) + 1 = \frac{-1 + 200}{200} = \boxed{\frac{199}{200} = L(.01)}$



what they're doing is

See $L(x)$ is ABOVE $f(x)$



3. SCalc8 2.9.005. (3354568)

Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$.

Use $L(x)$ to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$. (Round your answers to four decimal places.)

Illustrate by graphing f and the tangent line.

Look: To approximate $\sqrt{.9}$ and $\sqrt{.99}$, use the closest input that makes it clean: $x = 1$

Compare and contrast to the work done on the previous exercise, and how much harder it is to do, their way (for me, at least!).

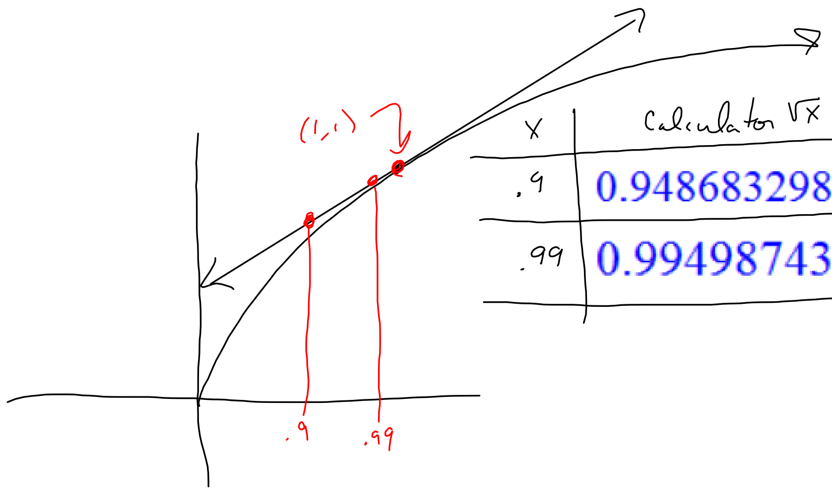
$f(x) = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}(x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$\Rightarrow f(a) = \sqrt{1} = 1 = y_1$ $\quad y = m(x - x_1) + y_1$

$\&\ f'(a) = \frac{1}{2\sqrt{1}} = \frac{1}{2} = m$ $\quad y = \frac{1}{2}(x-1) + 1 = L(x)$

$\Rightarrow \sqrt{.9} \approx \frac{1}{2}(.9 - 1) + 1 = \frac{1}{2}(-.1) + 1 = -\frac{1}{20} + \frac{20}{20} = \boxed{\frac{19}{10} \approx \sqrt{.9}}$

$\Rightarrow \sqrt{.99} \approx \frac{1}{2}(.99 - 1) + 1 = \frac{1}{2}(-.01) + 1 = -\frac{1}{200} + \frac{200}{200} = \boxed{\frac{199}{200} \approx \sqrt{.99}}$



X	Calculator $\sqrt{x}$	$L(x)$
.9	0.9486832981	0.9500000000
.99	0.9949874371	0.9950000000

4. SCalc8 2.9.006.MI. (3354495)

Find the linear approximation of the function $g(x) = \sqrt[3]{1+x}$ at $a = 0$.

Use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. (Round your answers to three decimal places.)

Illustrate by graphing g and the tangent line.

Use $g(x) = \sqrt[3]{x}$, $a = 1$, $x = .95$, $x = 1.1$

$\Delta x = -.05$, $\Delta = .1$

$g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$m(x - x_1) + y_1$

$g(1) = \sqrt[3]{1} = 1 = y_1$

$g'(1)(x-1) + g(1)$

$g'(x) = \frac{1}{3x^{2/3}} \Rightarrow g'(1) = \boxed{\frac{1}{3} = g'(1)}$

$\Rightarrow \boxed{L(x) = \frac{1}{3}(x-1) + 1}$

$L(.95) = \frac{1}{3}(.95 - 1) + 1$

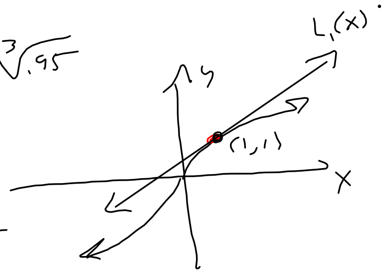
$= \frac{1}{3}(-.05) + 1$

$= \frac{-5}{3(100)} + \frac{300}{300} = \boxed{\frac{295}{300} = L(.95)} \approx \sqrt[3]{.95}$

$L(1.1) = \frac{1}{3}(1.1 - 1) + 1$

$= \frac{1}{3}\left(\frac{1}{10}\right) + 1$

$= \frac{1}{30} + \frac{30}{30} = \boxed{\frac{31}{30} = L(1.1)} \approx \sqrt[3]{1.1}$



5. SCalc8 2.9.007. (3354393)

Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1.

(Enter your answer using interval notation. Round your answers to three decimal places.)

$\sqrt[4]{1+2x} \approx 1 + \frac{1}{2}x$ $\boxed{(-0.368, 0.677)}$

$f(x) = (2x+1)^{\frac{1}{4}}$

$\Rightarrow f'(x) = \frac{1}{4}(2x+1)^{-\frac{3}{4}}(2) = \frac{1}{2}(2x+1)^{-\frac{3}{4}}$

$= \frac{1}{2\sqrt[4]{(2x+1)^3}}$

Want $|f(x) - L(x)| \leq .1$

$\left|\sqrt[4]{2x+1} - \frac{1}{2}x - 1\right| \leq .1$

$f(0) = \sqrt[4]{1} = 1$

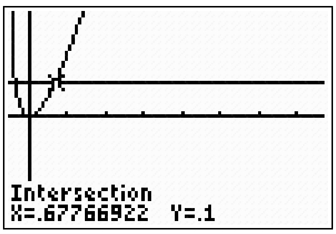
$-.1 \leq \sqrt[4]{2x+1} - \frac{1}{2}x - 1 \leq .1$

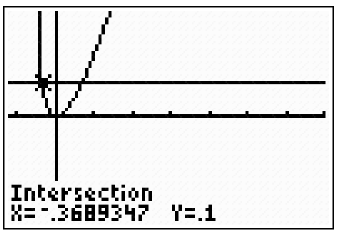
$f'(0) = \frac{1}{2(1)^{3/4}} = \frac{1}{2}$

$x \in (-0.369, 0.678)$ $y = L(x) = \frac{1}{2}(x - 0) + 1 = \frac{1}{2}x + 1$ ✓

& $(-.368, .677)$ is all very safe.

5.997257150





6. SCalc8 2.9.009. (3354336)

Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1.

(Enter your answer using interval notation. Round your answers to three decimal places.)

$\frac{1}{(1+3x)^4} \approx 1 - 12x$ $(-0.030, 0.037)$

Round down for "safety."

$(3x+1)^{-4} = f(x) \Longrightarrow$

$f'(x) = -4(3x+1)^{-5}(3) = \frac{-12}{(3x+1)^5}$.

$x_1 = a = 0 \Longrightarrow f'(a) = \frac{-12}{1^5} = -12 = m = f'(a)$

$y_1 = f(0) = \frac{1}{1^4} = 1 \Longrightarrow$

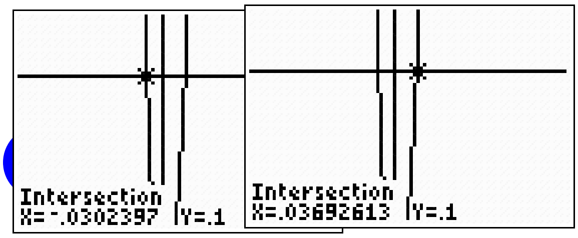
$y = L_0(x) = -12(x-0) + 1$

$y = f'(a)(x-a) + f(a)$

$y = m_{tan}(x - x_1) + y_1$

$\left| \frac{1}{(3x+1)^4} - 1 + 12x \right| < .1$

$-.1 < \frac{1}{(3x+1)^4} - 1 + 12x < .1$



$x \in (-0.030, 0.036)$

.036 is "safe." .037 is rounding as usual, & so I the book was off on #5.

Round As Usual.

7. SCalc8 2.9.013. (3354229)

Find the differential of each function.

(a) $y = \tan(\sqrt{3t})$

(b) $y = \frac{2 - v^2}{2 + v^2}$

The differential, $dy = f'(x)\,dx$
$= f'(x)\,\Delta x$

is an approximation for

$\Delta y = f(x_2) - f(x_1)$

$= y_2 - y_1 \approx m_{tan}(x - x_1)$

$m_{tan} = f'(x_1)$, $(x - x_1) = \Delta x = dx$!

8. SCalc8 2.9.015. (3354115)

(a) Find the differential *dy*.

$y = 3 \tan x$ $\quad$ $3\sec^2(x)\,dx$

(b) Evaluate *dy* for the given values of *x* and *dx*.

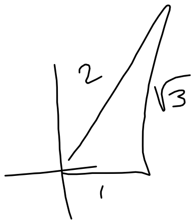
$x = \pi/3, \quad dx = -0.1$ $\quad$ -1.2

$$dy = f'(x)\,dx \approx \Delta y = f(x_2) - f(x_1)$$

$$\text{(a)} \quad \boxed{dy = (3\sec^2 x)\,dx}$$

$$dy\Big|_{\substack{x=\frac{\pi}{3}\\ dx=-.1}} = \left(3\sec^2\left(\tfrac{\pi}{3}\right)\right)(-.1) = 12(-.1) = \boxed{-1.2}$$

$$=$$



9. SCalc8 2.9.019. (3354118)

Compute Δy and dy for the given values of x and $dx = \Delta x$.

$y = x^2 - 4x, \quad x = 3, \quad \Delta x = 0.5$

1.25

1

Sketch a diagram showing the line segments with lengths dx, dy, and Δy.

$$x = 3, \quad \Delta x = dx = .5$$

$$x + \Delta x = 3.5$$

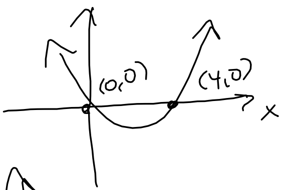
$$\Delta y = y_2 - y_2 = f(x+\Delta x) - f(x)$$

$$= f(3.5) - f(3)$$

$$= (3.5)^2 - 4(3.5) - (3^2 - 4(3))$$

$$= \boxed{1.25 = \Delta y}$$

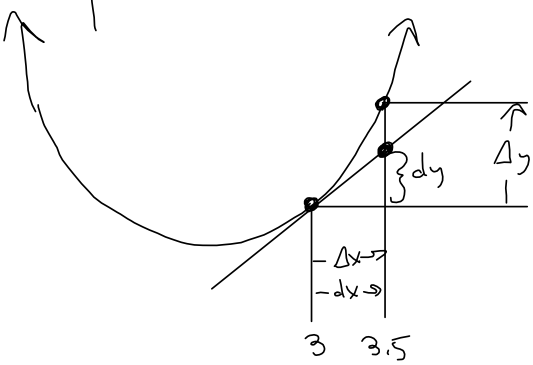
$$x^2 - 4x = x(x-4) = 0 \Rightarrow x \in \{0, 4\}$$



$$dy = f'(x)\,dx = (2x-4)\,dx$$

$$dy\Big|_{\substack{x=3\\ dx=.5}} = (2(3)-4)(.5)$$

$$= \underbrace{(2)}_{f'(3)}(.5) = \boxed{1 = dy}$$



10. SCalc8 2.9.023. (3354461)

Use a linear approximation (or differentials) to estimate the given number.

$(1.999)^3$

$f(x) = x^3$, $x_1 = 2$

$f(x_1) = f(2) = 2^3 = 8$

$(2, 8) = (x_1, y_1) = (x_1, f(x_1))$

$\Delta x = x_2 - x_1$

$= 1.999 - 2$

$= -.001$

$f'(x) = 3x^2$

$f'(2) = f'(x_1) = 3(2)^2 = 12 = f'(2)$

$y = m(x - x_1) + y_1$

$= f'(x_1)(x - x_1) + y_1$

$= f'(2)(x-2) + f(2)$

$= 12(x-2) + 8 \approx f(x) \approx L(x)$

$L(1.999) = 12\underbrace{(1.999-2)}_{\Delta x = dx} + 8$

$= 12(-.001) + 8$

$= -.012 + 8$

$= \boxed{7.988 = L(1.999) \approx 1.999^3}$

CALCULATOR

7.988005999

11. SCalc8 2.9.025. (3354203)

Question Content: view

Use a linear approximation (or differentials) to estimate the given number. (Round your answer to five decimal places.)

$\sqrt[3]{28}$ 3.03704

What's the closest value to $x = 28$ that has a nice, clean cube root?

Let $x_1 = 27$

Then $28 = 27 + 1 = x_1 + 1 = x_1 + \Delta x$

$\Rightarrow \Delta x = dx = 1$

$f(x) = x^{\frac{1}{3}} \Rightarrow$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$x = 27 \Rightarrow f(27) = \sqrt[3]{27} = 3$

$(x_1, f(x_1)) = (27, 3)$

$f'(x_1) = f'(27)$

$= \frac{1}{3}(27)^{-\frac{2}{3}} = \frac{1}{3(27^{1/3})^2}$

$\frac{1}{3(3^2)} = \frac{1}{27} = f'(x_1)$

$y \approx f'(x_1)(x - x_1) + f(x_1)$

$= \frac{1}{27}(x - 27) + 3 = L(x)$

$\Rightarrow L(28) = \frac{1}{27}(28 - 27) + 3 = \frac{1}{27} + \frac{81}{27}$

$= \frac{82}{27}$

$\approx$ 3.037037037

$f(28)$ directly

$\approx$ 3.036588972

12. SCalc8 2.9.031. (3354385)

The edge of a cube was found to be 15 cm with a possible error in measurement of 0.2 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing the volume of the cube and the surface area of the cube. (Round your answers to four decimal places.)

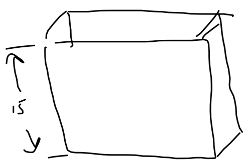
(a) the volume of the cube

maximum possible error	135	cm^3
relative error	0.0400	
percentage error	4.0000	%

(b) the surface area of the cube

maximum possible error	12d		36	cm^2
relative error	12e		0.0267	
percentage error	12f		2.6667	%

$15 \pm .2$ cm



$V = x^3$

$dV = 3x^2\,dx \approx \Delta V$

$\approx dV = 3(15)^2(\pm .2)$

$= \pm 3(225)(.2)$

$\boxed{dV = \pm 135 \text{ cm}^3 \approx \Delta V = \text{Error}}$

$\boxed{\text{Rel. Error} = \pm .04} = \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{135}{15^3} = .04$

Percent error: $(.04)(100\%)$

$\boxed{= \pm 4\%}$

(b) Surface Area $= 6x^2 = S(x)$

$dS = 12x\,dx$

$= 12(15)(\pm .2)$

$= \pm 180(.2)$

$\boxed{= \pm 36 \text{ cm}^2}$

$3 \cdot 225 \cdot 6 = 1350$

rel. error $\approx \frac{dS}{S} = \frac{\pm 36}{1350} = \frac{18}{775}$

$\boxed{\approx 0.02322580645 \approx \text{rel. error} \approx .0232}$

% error $\approx$ 2.322580645 %

$\approx$ $\boxed{2.3226\% \approx \%\text{ error}}$

13. SCalc8 2.9.032. (3354554)

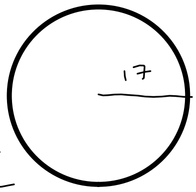
The radius of a circular disk is given as 17 cm with a maximum error in measurement of 0.2 cm.

(a) Use differentials to estimate the maximum error in the calculated area of the disk. (Round your answer to two decimal places.)

21.36 cm^2

(b) What is the relative error? (Round your answer to four decimal places.) 0.0235

What is the percentage error? (Round your answer to two decimal places.) 2.35 %



Area $= \pi r^2 = f$

$f(r_1) = f(17) = \pi(17)^2 = 289\pi$

$f'(r) = 2\pi r = \frac{df}{dr}$

$df = 2\pi r \, dr$

$= \pm 2\pi(17)(.2)$

$= \pm 6.8\pi \approx \boxed{\pm 21.36283005 \approx df \approx \pm 21.36}$

(b) $\frac{\Delta f}{f} \approx \frac{df}{f} \approx \frac{\pm 6.8\pi}{289\pi}$

$= \pm \frac{6.8}{289} \approx 0.02352941176$

$\approx \boxed{.0235 \approx \text{rel. error}}$

(c) $\boxed{\% \text{ error} \approx 2.35\ \%}$

14. SCalc8 2.9.034. (3354488)

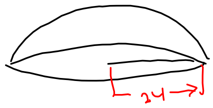
Use differentials to estimate the amount of paint needed to apply a coat of paint 0.03 cm thick to a hemispherical dome with diameter 48 m. (Round your answer to two decimal places.)

1.09 m^3

WANT $\Delta V = \frac{4}{3}\pi(34.03)^3 - \frac{4}{3}\pi(34)^3$ (2 … 2)

For SPHERE, so for hemisphere

$\frac{2}{3}\pi r^3$ is our volume



$V = \frac{2}{3}\pi r^3 \Rightarrow$

Mixing Units

$(.03cm)\left(\frac{1m}{100cm}\right) = .0003$

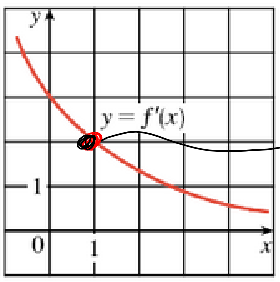
$\Delta V \approx dV = \frac{dV}{dr}dr = V'(r)dr$

$\frac{dV}{dr} = 2\pi r^2 \Rightarrow dV = 2\pi r^2 dr$

$= 2\pi(24)^2(.0003) \approx$ 1.085734421 m^3

15. SCalc8 2.9.041. (3354565)

Suppose that the only information we have about a function f is that $f(1) = -7$ and the graph of its *derivative* is as shown.



$f'(1) = 2$

$f(1) = -7$

$f'(1) = 2$

$f(x+\Delta x) \approx f(x) + f'(x)dx$

$y_2 \approx y_1 + f'(x_1)(x_2 - x_1)$

$L(x) = 2(x-1) - 7$

(a) Use a linear approximation to estimate $f(0.95)$ and $f(1.05)$.

$f(0.95) \approx$ -7.1

$f(1.05) \approx$ -6.9

$x + \Delta x = .95$

$1 + \Delta x = .95$

$\Delta x = -.05$

$L(.95) = 2(-.05) - 7$

$= -7.1$

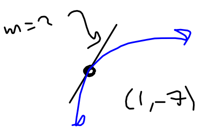
$= -7.1 \approx f(.95)$

$f(1.05) \approx L(1.05)$

$= 2(0.05) - 7$

$= -6.9 = L(1.05) \approx f(1.05)$

(b) Are your estimates in part (a) too large or too small? Explain.



Tangent Line Lies above $f(x)$

So, these are over estimates.

Diagram shows velocity $(f'(x))$ is decreasing & positive so steeper to left, less steep to the right

Concavity.

16. **SCalc8 2.9.JIT.001.MI.** (3389742)

Find an equation of the line that satisfies the given conditions.

Through (2, 2); slope 4 $\boxed{y = 4x - 6}$

$y = m(x - x_1) + y_1$ WAY BETTER THAN $y = mx + b$, for understanding

$y = 4(x-2)+2$ Stop!

$= 4x - 8 + 2$

$= 4x - 6$

17. **SCalc8 2.9.JIT.002.** (3389866)

Find the slope and y-intercept of the line and draw its graph. (If an answer does not exist, enter DNE.)

$x + y = 2$

x	y
0	2
2	0

slope $\boxed{-1}$

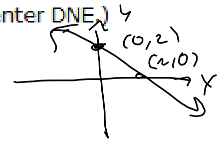
y-intercept $(x, y) = \boxed{(0, 2)}$

$x + y = 2$

$y = -x + 2$

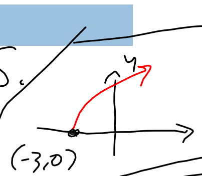
$\boxed{m = -1}$

y-int: $\boxed{(0,2)}$



18. **SCalc8 2.9.JIT.005.MI.** (3389832)

A graphing calculator is recommended. B.S.



A function f is given.

$f(x) = \sqrt{x + 3}$

(a) Use a graphing calculator to draw the graph of f.

(b) Find the domain and range of f from the graph.

Domain

$\mathcal{D} = [-3, \infty)$

$\mathcal{R} = [0, \infty)$

19. SCalc8 2.9.JIT.006.MI. (3389918)

A graphing calculator is recommended.

A function *f* is given.

$f(x) = -\sqrt{16 - x^2}$

(a) Use a graphing calculator to draw the graph of *f*.

(b) Find the domain and range of *f* from the graph.

$16 - x^2 \geq 0$

$(4+x)(4-x) \geq$

$-\quad +\quad -$

$-4 \quad 4$

$[-4, 4]$

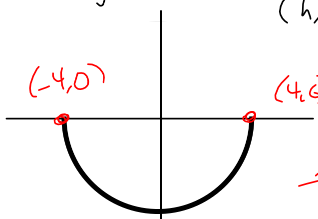
$y = -\sqrt{16 - x^2}$

$y^2 = 16 - x^2$

$x^2 + y^2 = 16$

$r = 4$

$(h, k) = (0, 0)$



$\mathcal{D} = [-4, 4]$

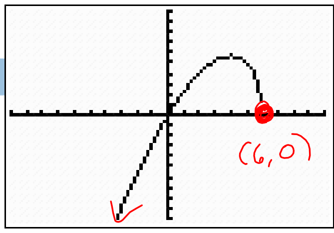
$\mathcal{R} = [-4, 0]$

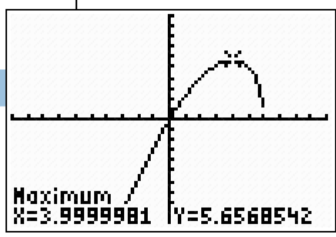
20. SCalc8 2.9.JIT.007. (3389957)

A graphing calculator is recommended.

The following function is given.

$U(x) = x\sqrt{6 - x}$





(a) Find all the local maximum and minimum values of the function and the value of *x* at which each occurs. State each answer correct to two decimal

local maximum (x, y) = (20a 4, 5.66)

$(4.00, 4.66)$

(b) Find the intervals on which the function is increasing and on which the function is decreasing. State each answer correct to two decimals.

Increasing: $(-\infty, 4]$ $(-\infty, 4)$

Decreasing: $[4, 6]$ $(4, 6)$