§2.6 Implicit Differentiation

Recall: Chain Rule:

$$\frac{d}{dx}\left[(f(x))^7\right] = 7f(x)^6 \cdot f'(x)$$

$$\frac{d}{dx}\left[xy^2\right] = 1y^2 + x \cdot 2y \cdot y' = y^2 + 2xyy'$$

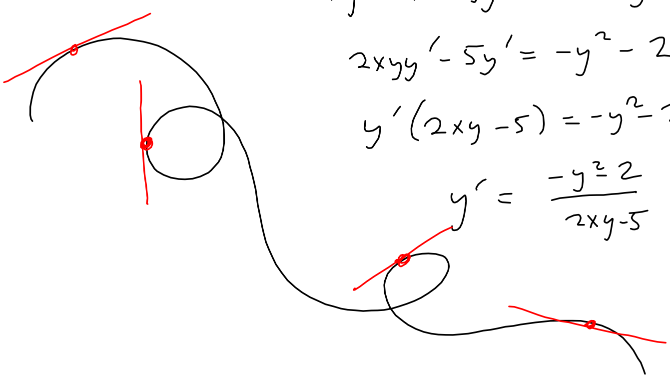
Find $\frac{dy}{dx}$: $xy^2 + 2x = 5y$

$$1y^2 + x \cdot 2yy' + 2 = 5y'$$

$$2xyy' - 5y' = -y^2 - 2$$

$$y'(2xy - 5) = -y^2 - 2$$

$$y' = \frac{-y^2 - 2}{2xy-5} = -\frac{y^2+2}{2xy-5} = \frac{dy}{dx}.$$



1. Question Details SCalc8 2.6.001.

Consider the following equation.

$2x^2 - y^2 = 3$ y is NOT f(x)

(a) Find y' by implicit differentiation.

(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.

(a) $4x - 2y \cdot y' = 0$

$-2yy' = -4x$

$y' = \frac{4x}{2y} = \frac{2x}{y}$

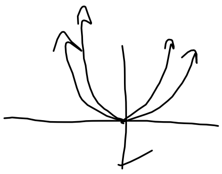
(b) $-y^2 = 3 - 2x^2$

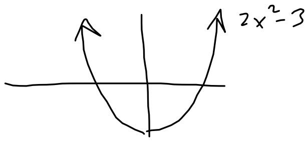
$y^2 = 2x^2 - 3$

$y = \pm\sqrt{2x^2-3} = \pm(2x^2-3)^{\frac{1}{2}}$

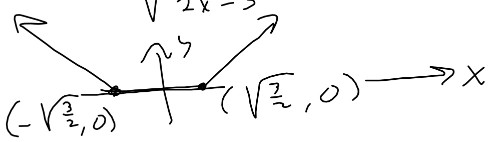
$y' = \pm\frac{1}{2}(2x^2-3)^{-\frac{1}{2}}(4x) = \boxed{\pm\frac{2x}{\sqrt{2x^2-3}}}$

$y = \pm\sqrt{2x^2-3}$

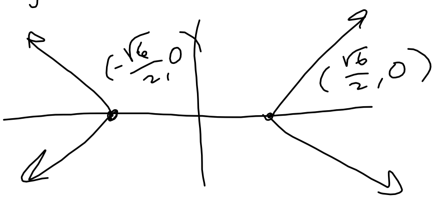




$\sqrt{2x^2-3}$



$y = \pm\sqrt{2x^2-3}$



2. Question Details SCalc8 2.6.005.

Find dy/dx by implicit differentiation.

$x^2 - 8xy + y^2 = 8$

Product Rule $(fg)' = f'g + fg'$

$\frac{d}{dx}[8xy] = 8y + 8xy'$

$f = 8x$

$g = y$

$f' = 1, \; g' = y'$

$2x - 8y - 8xy' + 2yy' = 0$

$y'(-8x + 2y) = -2x + 8y$

$$\boxed{y' = \frac{-2x+8y}{-8x+2y}}$$

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3. Question Details SCalc8 2.6.009.

Find dy/dx by implicit differentiation.

$$\frac{x^2}{x+y} = y^2 + 2$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\frac{(2x)(x+y) - x^2(1+y')}{(x+y)^2} = 2yy'$$

Thanks, Zach.

$$2x^2 + 2xy - x^2 - x^2y' = (2yy')(x+y)^2$$

$$+x^2y' + 2y(x+y)^2y' = +x^2 + 2xy$$

$$\boxed{y' = \frac{x^2+2xy}{x^2+2y(x+y)^2}}$$

$$\frac{d}{dx}[y^2] = 2y \cdot y'$$

4. Question Details SCalc8 2.6.014.

Find dy/dx by implicit differentiation.

$$3y\sin(x^2) = 7x\sin(y^2)$$

$$3y'\sin(x^2) + 3y\cos(x^2)\cdot 2x = 7\sin(y^2) + 7x\cos(y^2)\cdot 2yy'$$

$$3y'\sin(x^2) - 14xy\cos(y^2)\cdot y' = -6xy\cos(x^2) + 7\sin(y^2)$$

$$\boxed{y' = \frac{-6xy\cos(x^2) + 7\sin(y^2)}{3\sin(x^2) - 14xy\cos(y^2)}}$$

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5. Question Details SCalc8 2.6.020.

Find dy/dx by implicit differentiation.

$$\tan(x - y) = \frac{y}{5 + x^2}$$

Another from Zach

$$(\sec^2(x-y))(1-y') = \frac{(y')(x^2+5) - (y)(2x)}{(x^2+5)^2}$$

$$\sec^2(x-y) - y'\sec^2(x-y) = \frac{(x^2+5)y' - 2xy}{(x^2+5)^2}$$

$$(x^2+5)^2\sec^2(x-y) - (x^2+5)^2\sec^2(x-y)\,y' = (x^2+5)y' - 2xy$$

$$\left[+(x^2+5)^2\sec^2(x-y) + (x^2+5)\right]y' = +2xy + (x^2+5)^2\sec^2(x-y)$$

$$y' = \frac{2xy + (x^2+5)^2\sec^2(x-y)}{(x^2+5)^2\sec^2(x-y) + x^2+5}$$

6. Question Details SCalc8 2.6.023. [3354584]

Regard y as the independent variable and x as the dependent variable and use implicit differentiation to find dx/dy.

$$x^4y^2 - x^4y + 3xy^3 = 0$$

Now, instead of $y = f(x)$, it's $x = f(y)$

$4x^3 \cdot x' \cdot 2y$ Me saying $(fg)' = f'g'$ which is Bogus!

$$4x^3x'y^2 + x^4 \cdot 2y - 4x^3x'y - x^4 \cdot 1 + 3x'y^3 + 3x \cdot 3y^2 = 0$$

$$\Longrightarrow (4x^3y^2 - 4x^3y + 3y^3)x' = -2x^4y + x^4 - 9xy^2$$

$$\boxed{x' = \frac{-2x^4y + x^4 - 9xy^2}{4x^3y^2 - 4x^3y + 3y^3}}$$

7. Question Details SCalc8 2.6.024. [3354580]

Regard y as the independent variable and x as the dependent variable and use implicit differentiation to find dx/dy.

$y\sec(x) = 2x\tan(y)$ ugh $x = f(y)$, again.

$1\sec(x) + y\sec(x)\tan(x)\,x' = 2x'\tan(y) + 2x\sec^2(y)$

$(y\sec(x)\tan(x) - 2\tan(y))\,x' = -\sec(x) + 2x\sec^2(y)$

$$\boxed{x' = \frac{2x\sec^2(y) - \sec(x)}{y\sec(x)\tan(x) - 2\tan(y)}}$$

8. Question Details SC

Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

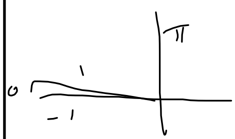
$y\sin(16x) = x\cos(2y)$, $(\pi/2, \pi/4)$ $y = f(x)$ $\frac{d}{dx}$ this one

$y'\sin(16x) + y\cos(16x)\cdot 16 = 1\cos(2y) - x\sin(2y)\cdot 2y'$

#8 INCORRECT

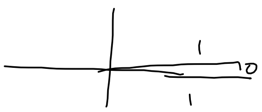
$y'\sin(16x) + 2x\sin(2y)y' = \cos(2y) - y\cos(16x)(16)$

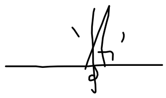
$\Rightarrow$ $y' = \left.\frac{\cos(2y) - y\cos(16x)}{\sin(16x) + 2x\sin(2y)}\right|_{(x,y)=(\frac{\pi}{2},\frac{\pi}{4})} = \frac{\cos(\frac{\pi}{2}) - \frac{\pi}{4}\cos(8\pi)\cdot(16)}{\sin(8\pi) + 2(\frac{\pi}{2})\sin(\frac{\pi}{2})}$ $(x,y) = (\frac{\pi}{2},\frac{\pi}{4})$



$= \frac{0 - \frac{\pi}{4}\cdot 16}{0 + \pi(1)} = \frac{(16)(-\frac{\pi}{4})}{\pi} = \frac{-\frac{16\pi}{4}}{\pi} = \frac{-16\pi}{4\pi} = -4$

$8\pi \leftrightarrow 2\pi \leftrightarrow 0$





$y = m(x - x_1) + y_1$

$= -\frac{1}{4}(x - \frac{\pi}{2}) + \frac{\pi}{4}$

8. Question Details SC

Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$y\sin(16x) = x\cos(2y), \quad (\pi/2, \pi/4)$ Charles.

$$y'\sin(16x) + y\cos(16x)\cdot 16 = 1\cos(2y) - x\sin(2y)\cdot 2y'$$

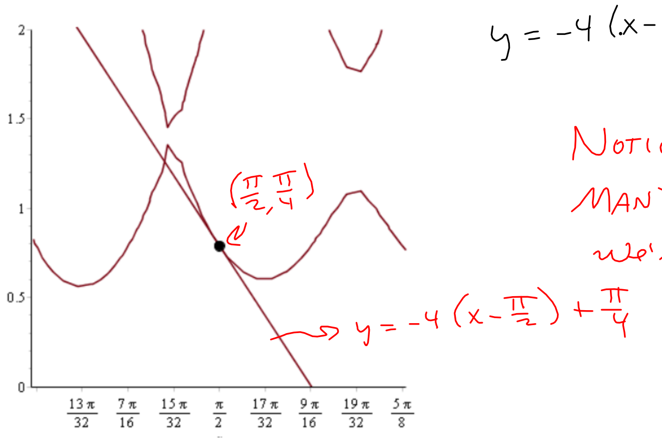
$$y'\sin(16x) + 2xy'\sin(2y) = \cos(2y) - 16y\cos(16x)$$

$$y' = \frac{\cos(2y) - 16y\cos(16x)}{\sin(16x) + 2x\sin(2y)}$$

$$y'\Big|_{(x,y)=(\frac{\pi}{2},\frac{\pi}{4})} = \frac{\cos(2\cdot\frac{\pi}{4}) - 16(\frac{\pi}{4})\cos(16\cdot\frac{\pi}{2})}{\sin(16\cdot\frac{\pi}{2}) + 2\cdot\frac{\pi}{2}\sin(2\cdot\frac{\pi}{4})}$$

$$= \frac{\cos(\frac{\pi}{2}) - 4\pi\cos(8\pi)}{\sin(8\pi) + \pi\sin(\frac{\pi}{2})} = \frac{0 - 4\pi(1)}{0 + \pi} = -4$$

$$y = -4(x - \tfrac{\pi}{2}) + \tfrac{\pi}{4}$$



NOTICE THERE ARE MANY POINTS $(\frac{\pi}{2}, y)$. we're just using the one, $(\frac{\pi}{2}, \frac{\pi}{4})$

9. Question Details SCalc8 2.6.030.

Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$x^{2/3} + y^{2/3} = 4$
$(-3\sqrt{3}, 1)$
(astroid)

$\Rightarrow \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0$

$\frac{2}{3}y^{-\frac{1}{3}}y' = -\frac{2}{3}x^{-\frac{1}{3}}$

$y' = -x^{-\frac{1}{3}}y^{\frac{1}{3}}$

$\Rightarrow y'\Big|_{(x,y)=(-3\sqrt{3},1)} = -(-3\sqrt{3})^{-\frac{1}{3}}(1) = \frac{1}{(3\sqrt{3})^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{3\sqrt{3}}} = \frac{1}{\sqrt[3]{3\cdot 3^{1/2}}} = \frac{1}{\sqrt{3}} \sqrt[6]{3}$

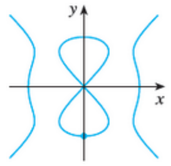
$\boxed{y = (3\sqrt{3})^{-\frac{1}{3}}(x+3\sqrt{3})+1}$

10. Question Details SCalc8 2.6.032.

Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$y^2(y^2 - 4) = x^2(x^2 - 5)$
$(0, -2)$
(devil's curve)

Guess : $\boxed{y=-2}$



$y^4 - 4y^2 = x^4 - 5x^2$

$4y^3y' - 8yy' = 4x^3 - 10x$

$y' = \frac{4x^3 - 10x}{4y^3 - 8y}$

$y'\Big|_{(x,y)=(0,-2)} = \frac{0-0}{4(-2)^3 - 8(-2)} = \frac{0}{-32+16} = 0$

$y = 0(x-0) - 2$

$y = -2$

$\boxed{y=-2}$

11. Question Details SCalc8 2.6.034. [3354466]

(a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, 2)$.

$y =$ ☐

(b) At what points does this curve have horizontal tangents?

(c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

(a) $2yy' = 3x^2 + 6x$

$y' = \frac{3x^2+6x}{2y}$

$\Rightarrow y'\Big|_{(x,y)=(1,2)} = \frac{3(1)^2+6(1)}{2(2)} = \frac{9}{4}$

$\boxed{y = \frac{9}{4}(x-1)+2}$

(b) $y' \overset{\text{SET}}{=} 0$

$3x^2+6x = 3x(x+2) = 0$

$\Rightarrow \boxed{x \in \{-2, 0\}}$

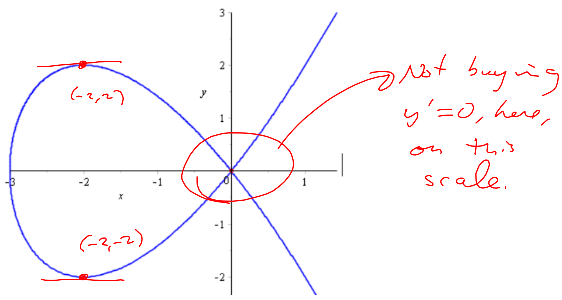
$x=-2$: $y^2 = x^3 + 3x^2$

$y^2 = (-2)^3 + 3(-2)^2$

$y^2 = -8 + 12 = 4$

$y = \pm 2 \rightarrow (-2,2), (-2,-2)$

$x=0 \Rightarrow y^2 = 0 \Rightarrow y = 0 \rightarrow (0,0)$



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12. Question Details — SCalc8 2.6.035. [3354181]

Find y'' by implicit differentiation.

$$x^2 + 6y^2 = 6$$

$$2x + 12yy' = 0$$

$$12yy' = -2x$$

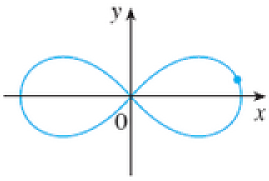
$$y' = -\frac{x}{6y}$$

$$y'' = -\left[\frac{(1)(6y) - x\cdot 6}{36y^2}\right] = \boxed{\frac{-6y + 6x}{36y^2} = y''}$$

13. Question Details — SCalc8 2.6.043. [3354155]

Find the points on the lemniscate where the tangent is horizontal.

$$2(x^2 + y^2)^2 = 81(x^2 - y^2)$$



$$4(x^2+y^2)(2x+2yy') = 81(2x-2yy')$$

$$4(2x^3+2x^2yy'+2xy^2+2y^3y' = 162x - 162yy'$$

$$2(2x^3+2x^2yy'+2xy^2+2y^3y') = 81x - 81yy'$$

$$4x^3+4x^2yy'+4xy^2+4y^3y' = 81x - 81yy'$$

$$4x^2yy'+4y^3y'+81yy' = 81x - 4x^3 - 4xy^2$$

$$y' = \frac{81x - 4x^3 - 4xy^2}{4x^2y + 81y + 4y^3} \quad \text{SET} \equiv 0$$

$$81x - 4x^3 - 4xy^2 = 0$$

$$x(81 - 4x^2 - 4y^2) = 0 \implies x = 0$$

$$-4x^2 - 4y^2 + 81 = 0$$

$$4x^2 + 4y^2 = 81$$

$$\text{circle}, \ r = \frac{9}{2}, \ (h,k) = (0,0)$$

14. Question Details — SCalc8 2.6.045. [3354239]

Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$$

$$-\frac{2yy'}{b^2} = -\frac{2x}{a^2}$$

$$y' = \frac{b^2x}{a^2y}$$

$$y = m(x - x_1) + y_1$$

$$= f'(x_1)(x - x_1) + y_1$$

$$= \frac{b^2x_1}{a^2y_1}(x - x_1) + y_1$$

15. Question Details SCalc8 2.6.047. [3354275]

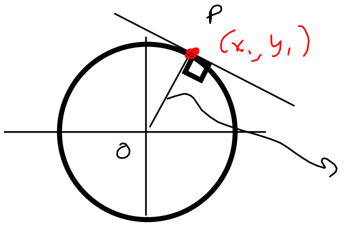
Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

$x^2+y^2=1$

$2x+2yy'=0$

$y'=-\frac{x}{y}$

$y=-\frac{x_1}{y_1}(x-x_1)+y_1$



$y=\frac{y_1}{x_1}x$

Slope of $\overline{OP}$ is $\frac{y_1}{x_1}$

$\frac{y_1}{x_1}=-\left(\frac{1}{\frac{x_1}{y_1}}\right)$

Slope of tangent.

Perp $m_\perp=-\frac{1}{m}$

So Perp.

16. Question Details SCalc8 2.6.051. [3354516]

Two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Are the given families of curves **orthogonal trajectories** of each other? That is, is every curve in one family orthogonal to every curve in the other family?

$y=cx^2,\ x^2+2y^2=k$

Sketch both families of curves on the same axes.

No

$y'=2cx$ parabolas

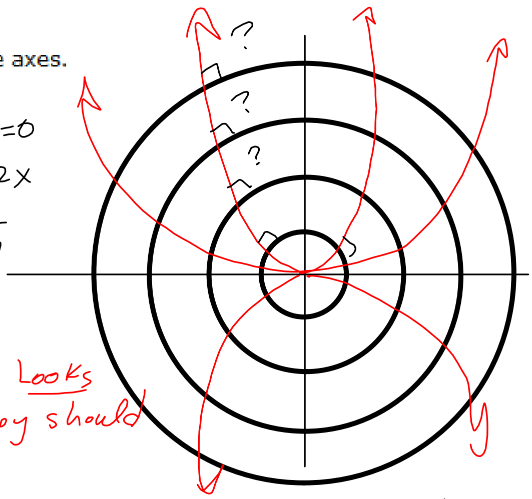
$2x+4yy'=0$

$4yy'=-2x$

$y'=-\frac{x}{2y}$

Not negative recips.

Almost Looks like they should be.



Where do cx^2 & $x^2+y^2=k$ intersect?

$cx^2=\sqrt{k-x^2}$

$y=\pm\sqrt{k-x^2} \rightarrow y=\sqrt{k-x^2}$

$c^2x^4=k-x^2$

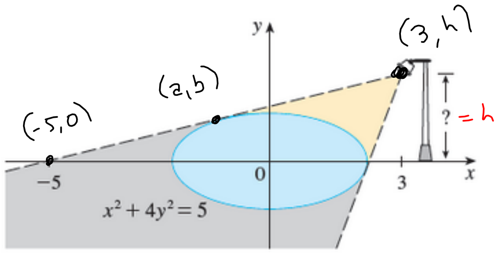
$c^2x^4+x^2-k=0$

will get back to you.

17. Question Details SCalc8 2.6.062. [3354469]

The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 \le 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?

units



$2x + 8yy' = 0$

$y' = -\frac{x}{4y}$

$m_{tan} = \frac{h-0}{3-(-5)} = \frac{h}{8} = \frac{1}{8}h$

$m_{tan} = -\frac{a}{4b} = \frac{1}{8}h = \frac{b}{a+5}$

$m_{tan} = \frac{b}{a+5} = -\frac{a}{4b}$

$\Rightarrow 4b^2 = -a^2 - 5a$

$\Rightarrow a^2 + 4b^2 = -5a = 5$, since on ellipse,

$\Rightarrow a = -1 \longrightarrow$

$(-1)^2 + 4b^2 = 5$

$4b^2 = 4$

$b^2 = 1$

$b = \pm 1 \longrightarrow b = 1$

$a = -1$
$b = 1$ & $-\frac{a}{4b} = \frac{1}{4} = \frac{1}{8}h$

$\boxed{2 = h}$