(15 pts) The point $P(2,-2)$ lies on the graph of $f(x)=x^2-3x$. Estimate the slope of this curve at $x=2$, by evaluating the average slope between P and the point $Q(x, x^2-3x)$, which is just another point on the graph of f. Use $x=2.001$ and $x=1.999$. I want your estimates to be accurate to the 4th decimal place.

$$m_{sec} = m_{avg} = \frac{f(2.001)-f(2)}{2.001-2} = \frac{2.001^2-3(2.001)-[2^2-3(2)]}{.001}$$

$$= \frac{4.004001-6.003-[-2]}{.001} = \frac{-1.998999+2}{.001}$$

$$= \frac{.001001}{.001} = \boxed{1.001} \quad \text{on } [2, 2.001]$$

on $[1.999, 2]$

$$\frac{f(1.999)-f(2)}{1.999-2} = \frac{1.999^2-3(1.999)-[-2]}{-.001}$$

$$= \frac{3.996001-5.997+2}{-.001} = \frac{-2.000999+2}{-.001}$$

$$\frac{-.000999}{-.001} = \boxed{.999}$$

(5 pts) Tell me what you think the precise slope of f is, at $x=2$. $m = 1$

(5 pts) Based on your answer to #2, write the equation of the tangent line to $f(x)=x^2-3x$ at $x=2$.

$$(x, f(x)) = (2,-2)$$

$$y = m(x-x_1)+y_1$$

$$y = 1(x-2)-2$$

4. (5 pts each) Evaluate the following limits, if they exist. When one does not exist, say so.

$$\lim_{x\to 2}\frac{x^2+5x-14}{2x^2-9x+10} = \lim_{x\to 2}\frac{(x+7)(x-2)}{(x-2)(2x-5)} = \lim_{x\to 2}\frac{x+7}{2x-5} = \frac{9}{-1} = -9$$

$$\begin{array}{r|rrr} 2 & 2 & -9 & 10 \\ & & 4 & -10 \\ \hline & 2 & -5 & \end{array}$$

$$\frac{x^2+5x-14}{2x^2-9x+10} = \frac{(x+7)(x-2)}{(x-2)(2x-5)} = \underset{(x\neq 2)}{\frac{x+7}{2x-5}} \xrightarrow{x\to 2} \frac{9}{-1} = \boxed{-9}$$

$$\lim_{x\to 5^-}\frac{|x-5|}{3x^2-11x-20} = \lim_{x\to 5^-}\frac{|x-5|}{(x-5)(3x+4)} = \lim_{x\to 5^-}\frac{-(x-5)}{(x-5)(3x+4)}$$

$$\begin{array}{r|rrr} 5 & 3 & -11 & -20 \\ & & 15 & 20 \\ \hline & 3 & 4 & \end{array}$$

$$= \lim_{x\to 5^-}\frac{-1}{3x+4} = \boxed{-\frac{1}{19}}$$

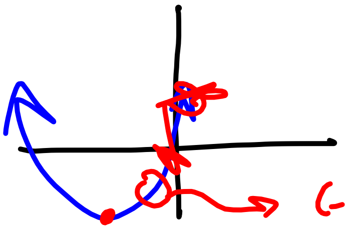
$$|x-5| = \begin{cases} x-5 & \text{if } x-5\geq 0 \\ -(x-5) & \text{if } x-5<0 \end{cases} = \begin{cases} x-5 & \text{if } x\geq 5 \\ -(x-5) & \text{if } x<5 \end{cases}$$

$\lim_{x\to 5}\frac{|x-5|}{3x^2-11x-20}$ Need to check $\lim_{x\to 5^+}$ (thing)

$$\lim_{x\to 5^+}\frac{|x-5|}{3x^2-11x-20} = \lim_{x\to 5^+}\frac{+(x-5)}{(x \ldots)} = \ldots = +\frac{1}{19}$$

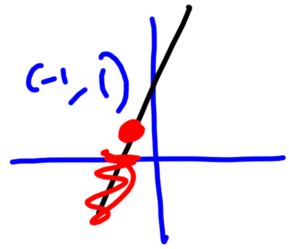
$\therefore$ $\lim_{x\to 5}\frac{|x-5|}{3x^2-11x-20}$ $\nexists$, b/c left- & right-hand limits disagree.

5. (15 pts) Sketch the graph of the piecewise-defined function $f(x)=\begin{cases}2(x+2)^2-3 & \text{if } x<-1\\ 2x+3 & \text{if } x\geq -1\end{cases}$. On what interval(s) is it continuous?



$(-1,-1)$

$(-2,-3)$

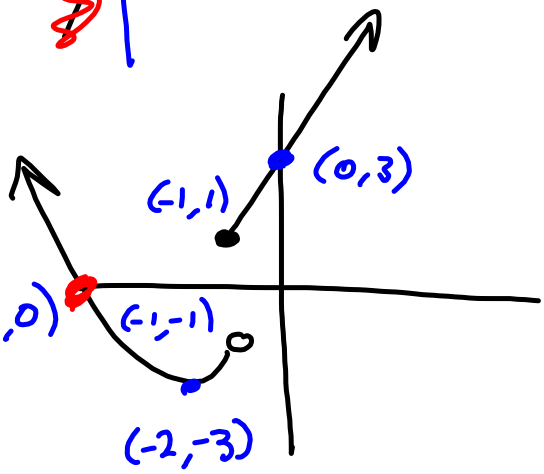


$(-1,1)$

$2(-1+2)^2-3$

$=2(1)^2-3$

$=-1$



$(-1,1)$ $(0,3)$ $\left(\frac{-4-\sqrt{6}}{2},0\right)$ $(-1,-1)$ $(-2,-3)$

$2(x+2)^2-3=0$

$2(x+2)^2=3$

$(x+2)^2=\frac{3}{2}$

$\sqrt{(x+2)^2}=\sqrt{\frac{3}{2}}$

$|x+2|=\sqrt{\frac{3}{2}}$

$x+2=\pm\sqrt{\frac{3}{2}}$

$x=-2\pm\sqrt{\frac{3}{2}}=-2\pm\frac{\sqrt{3}}{\sqrt{2}}=-2\pm\frac{\sqrt{6}}{2}$

$=\frac{-4\pm\sqrt{6}}{2}$

Bonus (5 pts) What value of a will make $f(x)=\begin{cases} 2(x+2)^2-3 & \text{if } x<-1 \\ 2x+a & \text{if } x\geq -1 \end{cases}$ continuous?

$2(x+2)^2-3=-1$ @ $x=-1$

want

$2x+a=-1$ @ $x=-1$

$2(-1)+a=-1$

$-2+a=-1$

$\boxed{a=1}$

$a=3$ made it $(-1,1)$

To hit $(-1,-1)$, subtract 2 from that:

$\boxed{a=1}$

6. Compute $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for the following.

a. (10 pts) $f(x) = x^2 - 3x - 7$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - 3(x+h) - 7 - [x^2 - 3x - 7]}{h}$$

$$= \frac{\cancel{x^2} + 2xh + h^2 \cancel{-3x} - 3h \cancel{-7} \cancel{-x^2} \cancel{+3x} \cancel{+7}}{h}$$

$$= \frac{2xh + h^2 - 3h}{h} = \frac{h(2x + h - 3)}{h} = 2x + h - 3$$

$(h \neq 0)$

$$\xrightarrow{h \to 0} \boxed{2x - 3}$$

b. (5 pts) $f(x)=\sqrt{x}$

$(a-b)(a+b) = a^2-b^2$

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$(h \neq 0)$

$$\xrightarrow{h \to 0} \frac{1}{\sqrt{x}+\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

7. (10 pts) Sketch a plausible graph of a function, f, that satisfies all of the properties listed.

a. $\lim_{x\to -5^-} f(x) = 3$

b. $\lim_{x\to -5^+} f(x) = 1$

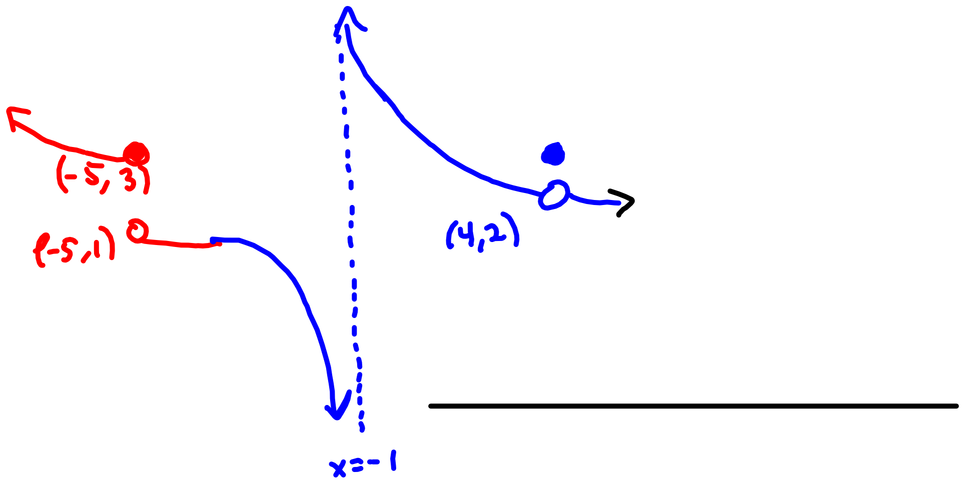
c. $f(-5) = 3$

d. $\lim_{x\to 1^-} f(x) = -\infty$

e. $\lim_{x\to 1^+} f(x) = \infty$

f. $\lim_{x\to 4} f(x) = 2$

g. $f(4) = 3$



8. (10 pts) Prove that $\lim_{x \to 2}(3x-2)=4$. (This is the $\varepsilon-\delta$ proof you're dying to do.)

want (scratch)

$|(3x-2)-4| < \varepsilon \Rightarrow$

$\Leftarrow |3x-6| < \varepsilon \Rightarrow$

$\Leftarrow 3|x-2| < \varepsilon \Rightarrow$

$\Leftarrow |x-2| < \frac{\varepsilon}{3} \equiv \delta$

Proof

Proof

Let $\varepsilon > 0$. Define $\delta = \frac{\varepsilon}{3}$.

$0 < |x-2| < \delta \Rightarrow$

$|(3x-2)-4| = |3x-6| = 3|x-2|$

$< 3\delta = 3 \cdot \frac{\varepsilon}{3} = \varepsilon$ ∎

"clearly"

9. (5 pts) Convince me that $f(x) = x^4 - 6x^3 + 2x^2 + 14x + 5$ has a zero in the interval $(2,3)$, without, you know, actually finding it. IVT

$$\begin{array}{r|rrrrr} 2 & 1 & -6 & 2 & 14 & 5 \\ & & 2 & -8 & -12 & 4 \\ \hline & 1 & -4 & -6 & 2 & \boxed{9 = f(2)} \end{array}$$

Remainder Theorem
College Algebra Videos
Test 3 Vid.

$$\begin{array}{r|rrrrr} 3 & 1 & -6 & 2 & 14 & 5 \\ & & 3 & -9 & -21 & -21 \\ \hline & 1 & -3 & -7 & -7 & \boxed{-16 = f(3)} \end{array}$$

$f(3) = -16 < 0 < 9 = f(2)$

$\Rightarrow \exists\, c \in (2,3) \ni$

$f(c) = 0$, by IVT.

(Polynomials are cnt$\underline{^{s}}$ on their domains.)

10. (10 pts) Prove that $\lim_{x\to 2}(x^2 - 3x) = -2$, using the precise definition of limit.

check $\boxed{2^2 - 3(2) = -2}$ ✓

$$\begin{array}{r|rrr} 2 & 1 & -3 & +2 \\ & & 2 & -2 \\ \hline & 1 & -1 & 0 \\ & x & c & r \end{array}$$

Want:

$|(x^2-3x) - (-2)| < \varepsilon$

$|x^2-3x+2| < \varepsilon$

$|(x-2)(x-1)| < \varepsilon$

$|x-1||x-2| < \varepsilon$

$< \delta$

Want a ceiling on $|x-1|$

Put a bound on x,

Restrict how far from $x=2$ we'll allow x to be.

$|x-2| < 1$

We're Assuming $\delta \le 1 \Longrightarrow$

$1 < x < 3$

$0 < x-1 < 2$

$|x-1| < 2$

Go Back

$|x-1||x-2| < 2|x-2|$ if $\delta \le 1$

want $2|x-2| < \varepsilon$

$|x-2| < \frac{\varepsilon}{2}$

Proof:

Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{2}\}$. Then

$0 < |x-2| < \delta \Longrightarrow |(x^2-3x)-(-2)| = |x^2-3x+2|$

$= |x-1||x-2| < 2|x-2| < 2\delta \le 2\cdot\frac{\varepsilon}{2} = \varepsilon$ ∎

11. (10 pts) Simplify the difference quotient and pass to the limit as h approaches zero for the function $f(x)=\frac{1}{\sqrt{x}}$.

$$\frac{f(x+h)-f(x)}{h}=\frac{1}{h}\left[f(x+h)-f(x)\right]$$

$$=\frac{1}{h}\left[\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right]=\frac{1}{h}\left[\frac{1}{\sqrt{x+h}}\cdot\frac{\sqrt{x}}{\sqrt{x}}-\frac{1}{\sqrt{x}}\cdot\frac{\sqrt{x+h}}{\sqrt{x+h}}\right]$$

LCD $=\sqrt{x}\sqrt{x+h}$

Rationalize the numerator

$$=\frac{1}{h}\left[\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right]$$

$(\sqrt{x}-\sqrt{x+h})$ $(\sqrt{x})^2=x$

$(a-b)(a+b)=a^2-b^2$

$(\sqrt{x+h})^2=x+h$

$$=\frac{1}{h}\left[\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right]\left[\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right]$$

$x-(x+h)$

$=x-x-h=-h$

$$=\frac{1}{h}\left[\frac{x-(x+h)}{(\sqrt{x}\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}\right]=\frac{1}{h}\left[\frac{-h}{(\sqrt{x}\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}\right]$$

$$=\frac{-1}{\sqrt{x}\sqrt{x+h}\,(\sqrt{x}+\sqrt{x+h})}\xrightarrow{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x}\,(\sqrt{x}+\sqrt{x})}$$

$(h\neq 0)$

$$=\frac{-1}{x\,(2\sqrt{x})}=-\frac{1}{2x\sqrt{x}}=-\frac{1}{2x^{3/2}}=\frac{-x^{-3/2}}{2}=\boxed{-\frac{1}{2}x^{-\frac{3}{2}}}$$

STOP

12. (10 pts) See if you can squeeze out a convincing argument for the claim $\lim_{x\to 0}\left(x^2\sin\left(\frac{1}{x}\right)\right)=0$.

$-1 \le \sin\left(\frac{1}{x}\right) \le 1$ any time $x \ne 0$.

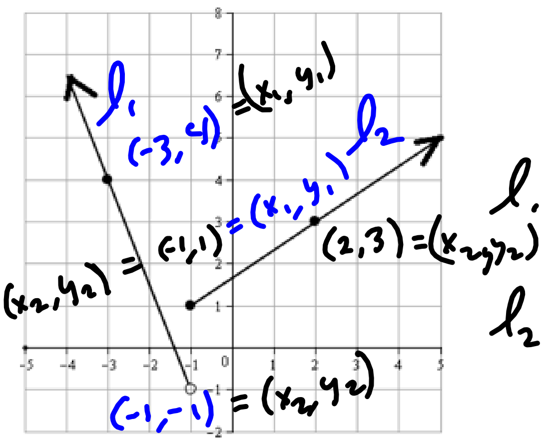
$$-x^2 \le x^2\sin\left(\frac{1}{x}\right) \le x^2$$

$$\underset{x\to 0}{\downarrow} \qquad \underset{x\to 0}{\downarrow} \qquad \underset{x\to 0}{\downarrow}$$

$$0 \le 0 \le 0$$

No Choice.

13. (10 pts) Write the definition of the piecewise-defined function from its graph.



l_1 : $x < -1$ $\quad$ $x=-1$ is "suture point"

l_2 : $x \geq -1$

l_1 : $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-1-4}{-1-(-3)} = \dfrac{-5}{2}$

l_2 : $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-1}{2-(-1)} = \dfrac{2}{3}$

$y = m(x - x_1) + y_1$

$$f(x) = \begin{cases} l_1 & \text{if } x < -1 \\ l_2 & \text{if } x \geq -1 \end{cases}$$

$$= \begin{cases} -\frac{5}{2}(x-(-3)) + 4 \\ \frac{2}{3}(x-(-1)) + 1 \end{cases}$$