2410 Midterm Solns Fall, 2025

① (a) 5pts $\lim_{x\to 2^-} \frac{x^2-7x+10}{|x-2|} = \lim_{x\to 2^-} \frac{(x-2)(x-5)}{-(x-2)} = \lim_{x\to 2^-} \frac{(x-5)}{-1} = \frac{2-5}{-1} = \boxed{3}$

(b) 5pts $\lim_{x\to 2^+} \frac{x^2-7x+10}{|x-2|} = \lim_{x\to 2^+} \frac{(x-2)(x-5)}{x-2} = \lim_{x\to 2} (2-5) = \boxed{-3}$

(c) 5pts $\lim_{x\to 2} \frac{x^2-7x+10}{x-2}$ $\nexists$ b/c $\lim_{x\to 2^+} \frac{x^2-7x+10}{|x-2|} = -3 \neq 3 = \lim_{x\to 2^-} \frac{x^2-7x+10}{|x-2|}$

② $f(x) = \begin{cases} x^2+3x-18 & \text{if } x<4 \\ 2x+2 & \text{if } x\geq 4 \end{cases}$

(a) 5pts

$x^2+3x-18 \quad x<4$

$= (x+6)(x-3) \leadsto (-6,0), (3,0)$

$h = \frac{-6+3}{2} = -\frac{3}{2}$

$k = (-\frac{3}{2})^2 + 3(-\frac{3}{2}) - 18$

$= \frac{9}{4} - \frac{9}{2} - 18 = \frac{9-18-72}{4} = \frac{9-90}{4} = \frac{-81}{4}$

$\lim_{x\to 4^-} f(x) = 4^2 + 3(4) - 18$

$= 16+12-18 = 10 \leadsto (4,10)$ o

$2x+2 \quad x\geq 4$

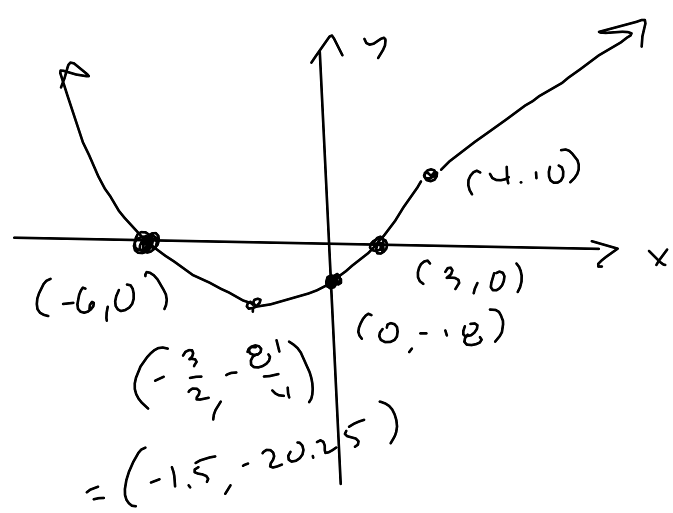
$(0,2), (-1,0)$

$\lim_{x\to 4^+} f(x) = 2(4)+2 = 10$

$\leadsto (4,10)$ •

Suture points the same!

$\Rightarrow$ cnts @ $x=4$



(b) 5pts f is cnts on $(-\infty,\infty)$, b/c each piece is a polynomial and they meet at the suture point.

(c) 5pts $f'(x) = \begin{cases} 2x+3 & x<4 \\ 2 & x>4 \end{cases}$

we check suture pt:

$f'(x) \xrightarrow{x\to 4^-} 2(4)+3 = 11 \neq 2 \Rightarrow$ Not difbl @ $x=4$.

2410 Midterm

(3) 5pts $f(x)=x^2-3x-5 \longrightarrow$

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-3(x+h)-5-(x^2-3x-5)}{h}$$

$$= \frac{x^2+2xh+h^2-3x-3h-\cancel{5}-x^2+3x+\cancel{5}}{h} = \frac{2xh+h^2-3h}{h} = \frac{h(2x+h-3)}{h}$$

$$\xrightarrow{h\to 0} \boxed{2x-3=f'(x)}$$

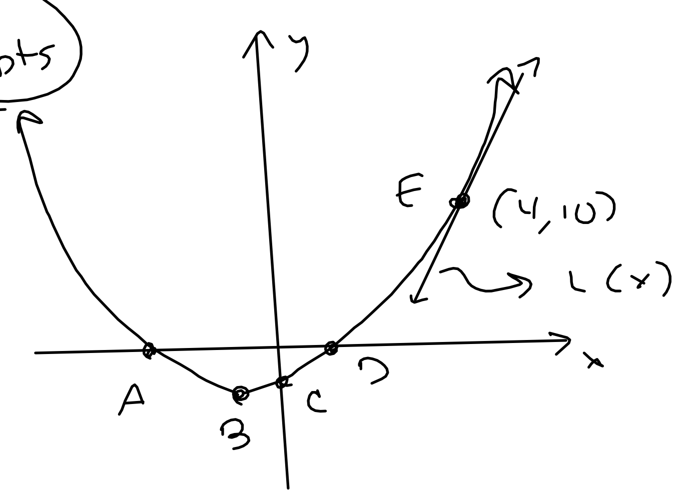
(4) See #2 for $x^2+3x-18$

$f'(x)=2x+3$

(a) 5pts $f'(4)=11=m_{tan}$

$$\boxed{L(x)=11(x-4)+10}$$

(b) 5pts



$A=(-6,0)$

$B=(-\frac{3}{2}, -\frac{81}{4})$

$C=(0,-18)$

$D=(3,0)$

$E=(4,10)$

(5) 5pts $\lim_{x\to 5}(2x-3)=7$

Proof

Let $\varepsilon>0$ be given. Define $\delta=\frac{\varepsilon}{2}$. Then $0<|x-5|<\delta \longrightarrow$

$|2x-3-7| = |2x-10| = 2|x-5| < 2\delta = 2\cdot\frac{\varepsilon}{2} = \varepsilon$ ∎

2410

midterm

6) 5pts $f(x) = x^4 - 9x^3 + 13x^2 + 9x - 14$ has a zero (root) in $(0,8)$.

$f(0) = -14$ & $f(8) = 378$

$f(0) = -14 < 0$ &

$f(8) = 378 > 0$.

8	1	-9	13	9	-14
		8	-8	40	392
	1	-1	5	49	378

f is poly. $\Rightarrow$ cnts $\Rightarrow$

Intermediate Value Theorem

hypotheses satisfied $\Rightarrow$

$\exists\, c \in (0,8) \ni f(c) = 0$

7) a) 5pts $f(x) = \sqrt[4]{x^3} - 5x^{\frac{4}{7}} - \frac{11}{x^{2/3}} = x^{\frac{3}{4}} - 5x^{\frac{4}{7}} - 11x^{-\frac{2}{3}}$

$$\Rightarrow \boxed{f'(x) = \frac{3}{4}x^{-\frac{1}{4}} - \frac{20}{7}x^{-\frac{3}{7}} + \frac{22}{3}x^{-\frac{5}{3}}}$$

b) 5pts $g(x) = \tan(4x)\csc(2x) \Rightarrow$

$$\boxed{g'(x) = 4\sec^2(4x)\csc(2x) + \tan(4x)\left(-2\csc(2x)\cot(2x)\right)}$$

c) 5pts $h(\theta) = \dfrac{\tan\theta}{\theta^3 + 2\theta}$ Poorly posed question 10pts correct 5pts Otherwise.

$$\Rightarrow \boxed{h'(\theta) = \frac{(\sec^2\theta)(\theta^3 + 2\theta) - (\tan\theta)(3\theta^2 + 2)}{(\theta^3 + 2\theta)^2}}$$

d) 5pts $r(w) = (w^2 + 11w + 5)^4(\cot(2w))^3 \Rightarrow$

$$\boxed{\begin{aligned} r'(w) = {} & 4(w^2 + 11w + 5)^3(2w + 11)(\cot(2w))^3 \\ & + (w^2 + 11w + 5)^4(3\cot^2(2w))(-\csc^2(2w))(2) \end{aligned}}$$

240

midterm

8 $\sin(x+y) = 2x - 2y \Rightarrow$

a 5pts $(\cos(x+y))(1+y') = 2-2y'$

$\Rightarrow \cos(x+y) + y'\cos(x+y) = 2-2y'$

$\Rightarrow y'\cos(x+y) + 2y' = 2 - \cos(x+y)$

$\Rightarrow \boxed{y' = \dfrac{2-\cos(x+y)}{\cos(x+y)+2}}$

b 5pts $y'\Big|_{(x,y)=(\pi,\pi)} = \dfrac{2-\cos(\pi+\pi)}{\cos(\pi+\pi)+2} = \dfrac{2-1}{1+2} = \dfrac{1}{3} = m$

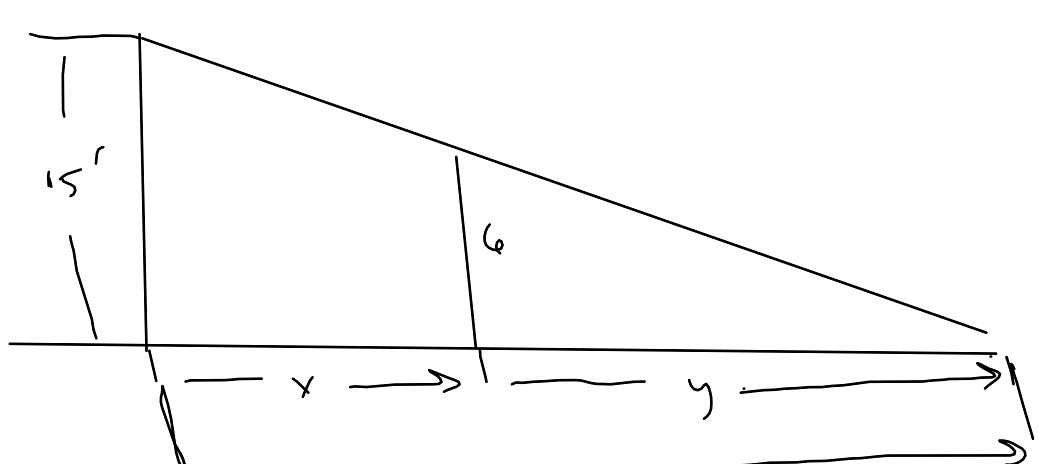
$y = L(x) = m(x-x_1)+y_1 = \boxed{\frac{1}{3}(x-\pi)+\pi = L(x)}$

$= \frac{1}{3}x - \frac{\pi}{3} + \frac{3\pi}{3} = \frac{1}{3}x + \frac{2\pi}{3}$

9 10pts Let x = distance from man to pole (in feet)

Let y = distance from man to tip of his shadow (ft)

Let z = " " pole to " " " " (ft)



Question asks for $\left.\dfrac{dz}{dt}\right|_{x=40}$

$z = x+y$

$\dfrac{dz}{dt} = \dfrac{dx}{dt} + \dfrac{dy}{dt}$

Given $\dfrac{dx}{dt} = 5\text{ ft/s}$

Similar triangles:

$\dfrac{15}{z} = \dfrac{6}{y} \Rightarrow 15y = 6z$

$\Rightarrow y = \dfrac{6}{15}z = \dfrac{3}{5}z$

$\Rightarrow z = x + \dfrac{3}{5}z$

$\Rightarrow \dfrac{2}{5}z = x \Rightarrow$

$\dfrac{2}{5}\dfrac{dz}{dt} = \dfrac{dx}{dt} = 5 \Rightarrow \dfrac{dz}{dt} = \boxed{\dfrac{25}{2}\text{ ft/s} = \dfrac{dz}{dt}}$

2410

Midterm

10 (10 pts) Minute hand is r cm. Find the rate at which it sweeps out area as a function of r.

Area $= \frac{1}{2}r^2\theta = A$

We know the minute hand sweeps out one revolution per hour.

$$\left(\frac{1 \text{ rev}}{hr}\right)\left(\frac{2\pi \text{ radians}}{1 \text{ rev}}\right) \Rightarrow \frac{d\theta}{dt} = \frac{2\pi}{hr}$$

$$\frac{dA}{dt} = 2\left(\frac{1}{2}\right)r\frac{dr}{dt} + \frac{1}{2}r^2\frac{d\theta}{dt}$$

$$= 0 + \frac{1}{2}r^2\left(\frac{2\pi}{hr}\right) = \boxed{\pi r^2 \ \frac{cm^2}{hr}}$$

11 Radius of sphere is $2 \text{ cm} \pm 0.1 \text{ cm}$

a (5pts) $A = $ Area $(cm^2) = 4\pi r^2$, where $r = $ radius

$dr = \Delta r = \pm 0.1 \text{ cm}$

Error in area $= \Delta A \approx dA = 8\pi r\,dr = 8\pi(2)(0.1) = 16(.1)\pi$

$= \boxed{1.6\pi \ cm^2} \approx \boxed{5.026548246 \ cm^2}$

b (5pts) $V = $ Volume $= \frac{4}{3}\pi r^3 \Rightarrow$

Error in volume is

$\Delta A \approx dA = 4\pi r^2 dr = 4\pi(2)^2(0.1) = \boxed{1.6\pi \approx 5.026548246 \ cm^3}$

2410

Midterm
BONUS

(1) (5pts) $\lim_{x \to 2} (2x^2 - 3x - 1) = \lim f = 1$. Let $\varepsilon > 0$ be given.

Want $|f(x) - L| = |2x^2 - 3x - 1 - 1| = |2x^2 - 3x - 2| = |2x+1||x-2|$
$< |2x+1| \delta < \varepsilon$. Assume $0 < |x-2| < \delta$

Assume $\delta \leq 1$. Then $|x-2| < \delta$ means

$$1 < x < 3$$
$$\Rightarrow \quad 2 < 2x < 6$$
$$3 < 2x+1 < 7$$
$$\Rightarrow |2x+1| < 7$$

Proof Let $\varepsilon > 0$. Define $\delta = \min\{1, \frac{\varepsilon}{7}\}$. Then $0 < |x-2| < \delta$ implies $|f(x) - L| = |2x^2 - 3x - 1 - 1| = |2x^2 - 3x - 2| = |2x+1||x-2|$
$< 7\delta \leq 7 \cdot \frac{\varepsilon}{7} = \varepsilon$ ■

(2) (5pts) $f(x) = x^{\frac{1}{3}} \Rightarrow$

$$\frac{f(x) - f(c)}{x - c} = \frac{x^{\frac{1}{3}} - c^{\frac{1}{3}}}{x - c} = \frac{(x^{\frac{1}{3}} - c^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}})}{(x-c)(x^{\frac{2}{3}} + x^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}})}$$

$$= \frac{x - c}{(x-c)(x^{\frac{2}{3}} + (xc)^{\frac{1}{3}} + c^{\frac{2}{3}})} = \frac{1}{x^{\frac{2}{3}} + (xc)^{\frac{1}{3}} + c^{\frac{2}{3}}} \xrightarrow{x \to c}$$

$(x \neq c)$

$$\frac{1}{c^{\frac{2}{3}} + (c^2)^{\frac{1}{3}} + c^{\frac{2}{3}}} = \frac{1}{3c^{\frac{2}{3}}} = \boxed{\frac{1}{3} c^{-\frac{2}{3}} = f'(c)}$$

2410

MIDTERM

3 5pts $f(x)=\begin{cases} x^2\sin\left(\frac{\pi}{x}\right) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

f is the product of a polynomial and a trig function of a rational function away from $x=0$, and is cnt^s on that domain. It remains to be seen whether

$$\lim_{x\to 0} f(x) = f(0) = 0.$$

FACT: $-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$

$$\Rightarrow -x^2 \leq x^2\sin\left(\frac{\pi}{x}\right) \leq x^2$$

$$0 \leq \lim_{x\to 0} x^2\sin\left(\frac{\pi}{x}\right) \leq 0 \Rightarrow \lim_{x\to 0} x^2\sin\left(\frac{\pi}{x}\right) = 0$$

by the Squeeze Theorem $\Rightarrow \lim_{x\to 0} f(x) = f(0) \Rightarrow$

f is cnt^s EVERYWHERE. ∎