1 a) 5pts $f(x) = x^2 + 4x \longrightarrow$

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 + 4(x+h) - (x^2+4x)}{h}$$

$$= \frac{x^2+2hx+h^2+4x+4h-x^2-4x}{h} = \frac{2xh+h^2+4h}{h}$$

$$= \frac{h(2x+h+4)}{h} \xrightarrow{h \to 0} \boxed{2x+4 = f'(x)}$$

b) 5pts $\int_{-2}^{2} f(x)\,dx \qquad \Delta x = \frac{b-a}{n} = \frac{2-(-2)}{n} = \frac{4}{n}$

Note $[a,b] = [-2,2]$ & x is odd $\Rightarrow \int_{-2}^{2} 4x\,dx = 0.$! so we may

$x_k = a + k\Delta x = -2 + \frac{4}{n}k = \frac{4k}{n} - 2$

safely ignore $4x$ in the discussion!

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}(x_k^2)\cdot\frac{4}{n} = \frac{4}{n}\sum_{k=1}^{n}\left(\frac{4k}{n}-2\right)^2$$

$$= \frac{4}{n}\sum_{k=1}^{n}\left(\frac{16k^2}{n^2} - \frac{16k}{n} + 4\right)$$

$$= \frac{4}{n}\cdot\frac{16}{n^2}\sum_{k=1}^{n}k^2 - \frac{4}{n}\cdot\frac{16}{n}\sum_{k=1}^{n}k + \frac{4}{n}\sum_{k=1}^{n}4$$

$$= \frac{64}{n^3}\left(\frac{n^3+\sim}{3}\right) - \frac{64}{n^2}\left(\frac{k^2+\sim}{2}\right) + \frac{16}{n}\cdot n$$

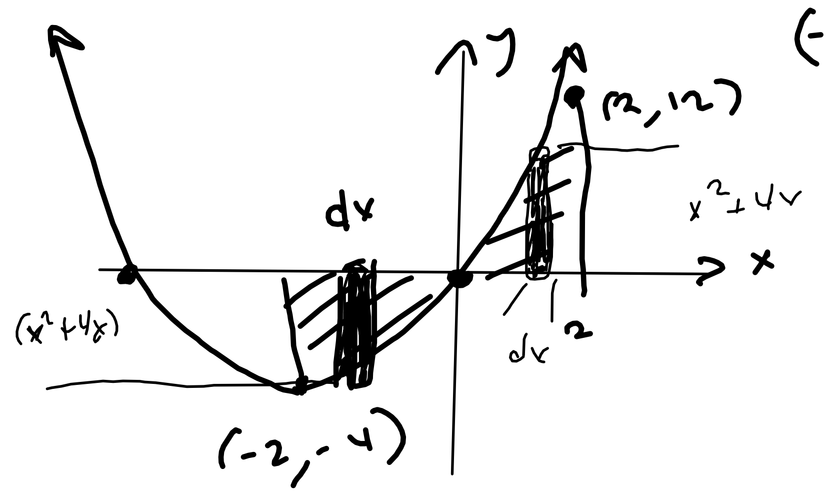
$$\xrightarrow{n \to \infty} \frac{64}{3} - \frac{64}{2} + 16 = \frac{64}{3} - 32 + 16$$

$$= \frac{64}{3} - 16\cdot\frac{3}{3} = \frac{64-48}{3} = \boxed{\frac{16}{3} = \int_{-2}^{2}(x^2+4x)\,dx}$$

2 a 5pts

$x^2+4x=0 \Rightarrow x=0,-4$

$(-2)^2+4(-2) = 4-8=-4$



b 5pts Area $= \int_{-2}^{0} (0-(x^2+4x))\,dx + \int_{0}^{2} (x^2+4x)\,dx$

c 5pts $= \left[-\frac{x^3}{3}-\frac{4x^2}{2}\right]_{-2}^{0} + \left[\frac{x^3}{3}+2x^2\right]_{0}^{2}$

$= 0-\left(-\frac{(-2)^3}{3}-\frac{4(-2)^2}{2}\right) + \left[\frac{2^3}{3}+2(2^2)\right]$

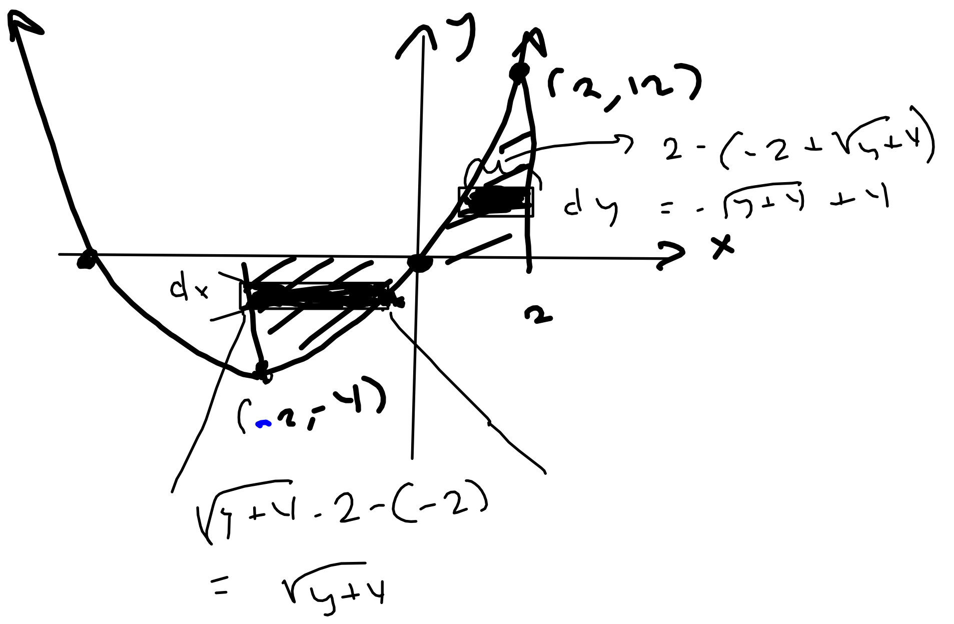
$= -\left(-\left(-\frac{8}{3}\right)-\frac{4(4)}{2}\right)+\frac{8}{3}+8$

$= -\left(\frac{8}{3}-8\right)+\frac{8}{3}+8 = 8-\frac{8}{3}+\frac{8}{3}+8 = \boxed{16=\int_{-2}^{2}|f(x)|\,dx}$

d 5pts Not the same. #1b integral is SIGNED area.

#2c is just area.

(e) 5pts Re-sketch wrt "y" variable



$x^2+4x=y$

$x^2+4x+2^2=y+4$

$(x+2)^2=y+4$

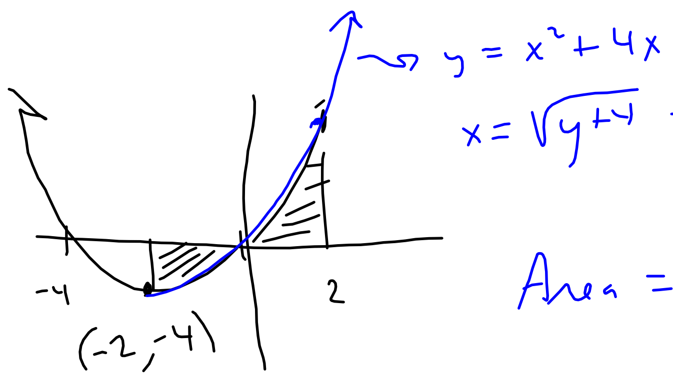
$x+2=\pm\sqrt{y+4}$

$x=-2+\sqrt{y+4}$

choose "+", b/c $x \geq -2$

(f) 5pts see rectangles above

Area $= \int^{0} \sqrt{y+4}\,dy + \int_0^{12} (4-\sqrt{y+4})\,dy$



$x=\sqrt{y+4}-2 = g(y) = -2+\sqrt{y+4}$

$$\text{Area} = \int_{-4}^{0} (g(y)-(-2))\,dy + \int_0^{12} (2-g(y))\,dy$$

$$= \int_{-4}^{0} (-2+\sqrt{y+4}+2)\,dy + \int_0^{12} (2-(-2+\sqrt{y+4}))\,dy$$

$$= \int_{-4}^{0} (y+4)^{\frac{1}{2}}\,dy + \int_0^{12} (4-(y+4)^{\frac{1}{2}})\,dy$$

$$= \left.\frac{(y+4)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-4}^{0} + \left[4y-\frac{(y+4)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{12}$$

$$= \frac{2}{3}\left[4^{\frac{3}{2}}-0^{\frac{3}{2}}\right] + \left[4(12)-\frac{2}{3}(16)^{\frac{3}{2}} - \left(4(0)-\frac{2}{3}(4)^{\frac{3}{2}}\right)\right]$$

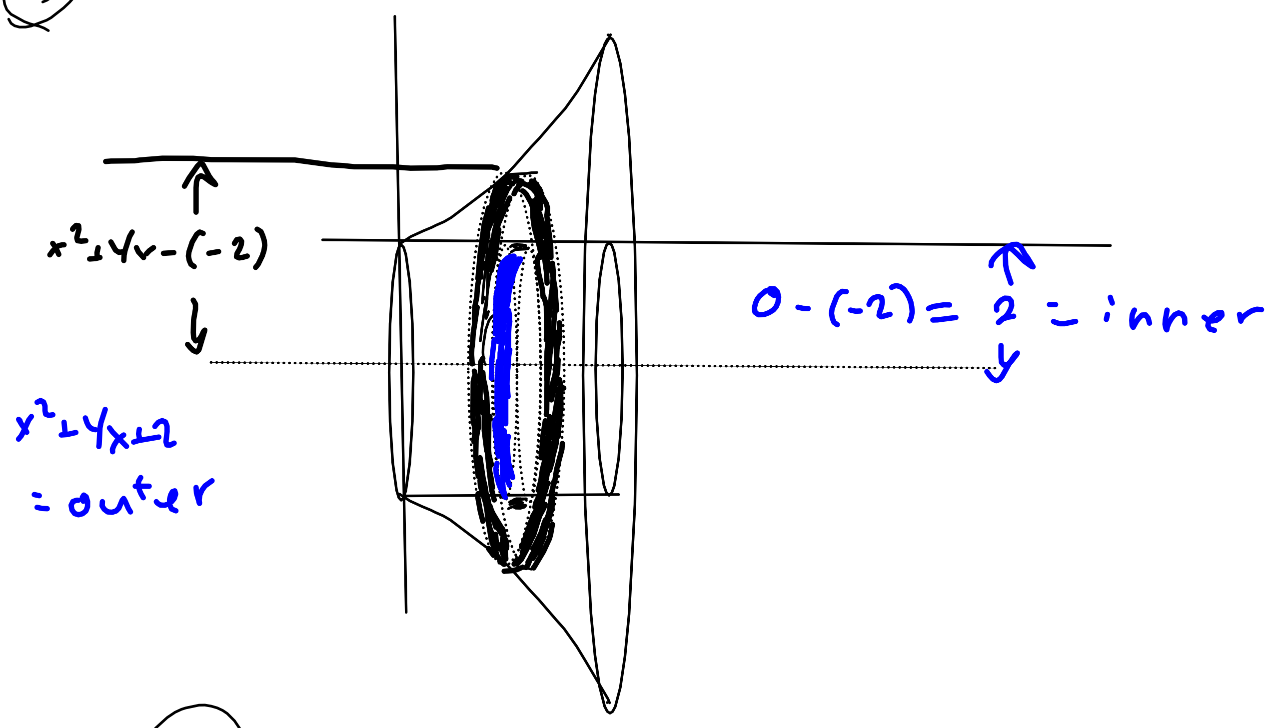
$$= \frac{2}{3}(2^3) + \left[48-\frac{2}{3}(4)^3 - \left(-\frac{2}{3}(2)^3\right)\right]$$

$$= \frac{2}{3}(8)+48-\frac{2}{3}(64)+\frac{2}{3}(8)$$

$$= \frac{16}{3}-\frac{128}{3}+\frac{16}{3}+48\cdot\frac{3}{3} = \frac{128-32}{3}+48 = -\frac{96}{3}+48 = -32+48 = 16$$

Area $= 16$

3 (a) sketch 5pts



(b) 5pts Volume $= \pi \int_0^2 \left(\left(x^2+4x+2\right)^2 - \left(2^2\right)\right) dx$

$(x^2+4x+2)(x^2+4x+2)$

$= x^4+4x^3+2x^2$
$\quad + 4x^3+16x^2+8x$
$\quad + 2x^2+8x+4$

$x^4+8x^3+20x^2+16x+4$

(c)

$= \pi \int_0^2 (x^4+8x^3+20x^2+16x+4-4)\, dx$

$= \pi \left[\frac{1}{5}x^5 + 2x^4 + \frac{20}{3}x^3 + 8x^2 \right]_0^2$

$= \pi \left[\frac{1}{5}(32) + 32 + \frac{160}{3} + 32 \right]$

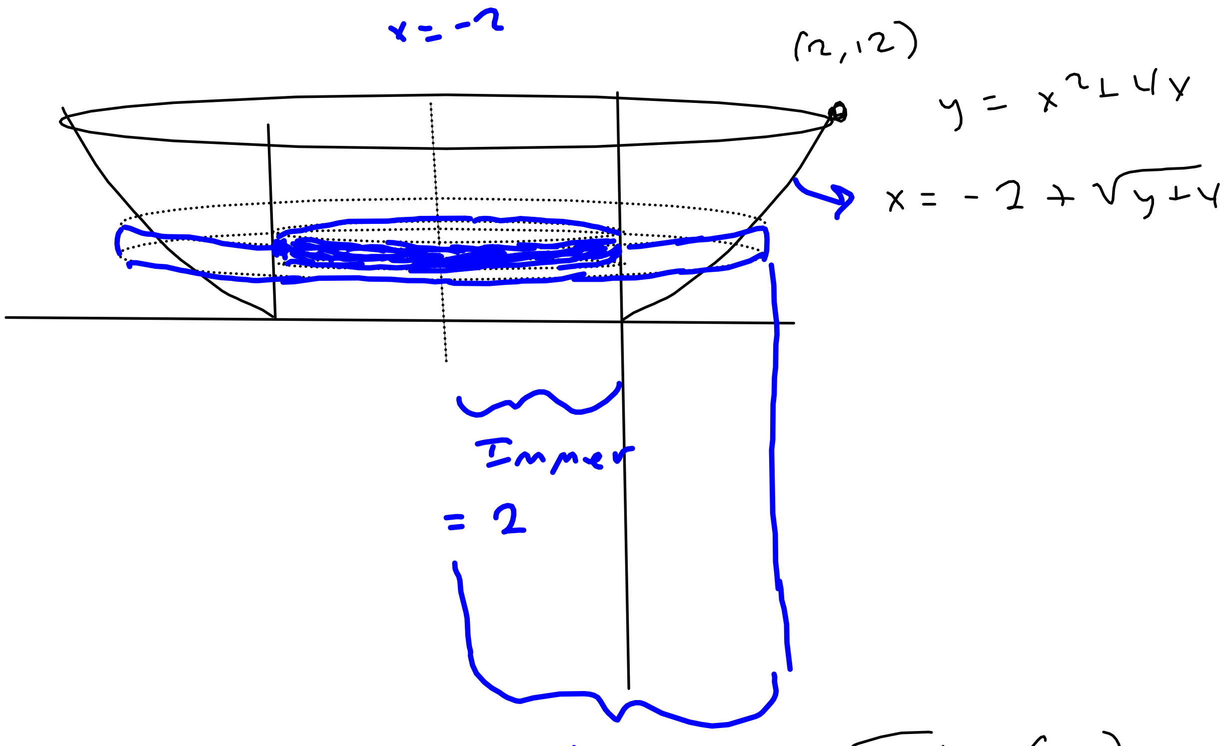
$= \pi \left[\frac{32}{5} \cdot \frac{3}{3} + \frac{160}{3} \cdot \frac{5}{5} + 64 \right]$

640
320
960

$= \pi \left[\frac{96+800}{15} + 64 \right] = \pi \left[\frac{896}{15} + \frac{64}{1} \cdot \frac{15}{15} \right]$

$= \pi \left[\frac{896+960}{15} \right] = \boxed{\frac{1856\pi}{15}}$

3 d 5pts



outer $= -2 + \sqrt{y+4} - (-2)$

$= \sqrt{y+4}$

d 5pts

$$\text{Volume} = \pi \int_c^d (\text{outer}^2 - \text{inner}^2)\, dy$$

e 5pts

$$= \pi \int_0^{12} \left((\sqrt{y+4})^2 - (2^2) \right) dy = \pi \int_0^{12} (y + 4 - 4)\, dy =$$

$$= \pi \left[\frac{1}{2} y^2 \right]_0^{12} = \pi\left(\frac{1}{2}(144)\right) = \pi(72) = \boxed{72\pi = \text{Volume}}$$

4 a 5pts $y = 5 \cdot 2^{\sin(x)} \longrightarrow \boxed{y' = 5\cos(x)\ln(2) \cdot 2^{\sin(x)}}$

b 5pts $y = \ln\left(\frac{(x^2-7x)^2}{\sqrt[3]{\sin^2(x)}}\right) = 2\ln(x^2-7x) - \frac{2}{3}\ln(\sin(x)) \longrightarrow$

$$\boxed{y' = 2\left(\frac{2x-2}{x^2-2x}\right) - \frac{2}{3}\left(\frac{\cos(x)}{\sin(x)}\right)}$$

c 5pts $y = \log_7(x\sin(5x)) = \log_7(x) + \log_7(\sin(5x))$

$$\longrightarrow \boxed{y' = \frac{1}{\ln(7)}\left(\frac{1}{x}\right) + \frac{1}{\ln(7)}\left(\frac{5\cos(5x)}{\sin(5x)}\right)}$$

d 5pts $y = [x^2+5x]^{\tan(x)} \longrightarrow$

$\ln(y) = \tan(x)\ln(x^2+5x) \longrightarrow$

$$\frac{y'}{y} = \sec^2(x)\ln(x^2+5x) + \tan(x)\left(\frac{2x+5}{x^2+5x}\right)$$

$$\longrightarrow \boxed{y' = \left(\sec^2(x)\ln(x^2+5x) + \tan(x)\left(\frac{2x+5}{x^2+5x}\right)\right)[x^2+5x]^{\tan(x)}}$$

4 Cont'd

e $y = \int_0^x \frac{(t^2-7t)^2}{\sqrt[3]{\sin^2(t)}}\,dt \longrightarrow$

$$\boxed{y' = \frac{(x^2-7x)^2}{\sqrt[3]{\sin^2(t)}}}$$

f 5pts $y = \int_0^{\tan(x)} \frac{(t^2-7t)^2}{\sqrt[3]{\sin^2(t)}}\,dt \longrightarrow$

$$\boxed{y' = \frac{(\tan^2(x)-7\tan(x))^2}{\sqrt[3]{\sin^2(\tan(x))}} \cdot \sec^2(t)}$$

5 (a) 5pts $\int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C$

$\boxed{= \ln|\sec(x)| + C}$

$u = \cos(x)$

$du = -\sin(x)\,dx$

(b) 5pts $\int \frac{\cos(x)}{\sin^5(x)}\,dx = \int \frac{du}{u^5} = \int u^{-5}\,du = \frac{1}{-4}u^{-4} + C = \boxed{-\frac{1}{4}\sin^{-4}(x) + C}$

$u = \sin(x)$

$du = \cos(x)\,dx$

(c) 5pts $\int x^3(3x-5)^7\,dx =$

$u = 3x-5$

$du = 3\,dx$

$\frac{du}{3} = dx$

$3x-5 = u$

$3x = u+5$

$x = \frac{u+5}{3}$

$= \int \left(\frac{u+5}{3}\right)^3 u^7 = \frac{1}{27}\int u^7\left(u^3 + 3u^2\cdot 5 + 3u\cdot 5^2 + 5^3\right)du$

$= \frac{1}{27}\int \left(u^{10} + 15u^9 + 75u^8 + 125u^7\right)du$

$\boxed{= \frac{1}{27}\left[\frac{1}{11}(3x-5)^{11} + \frac{75}{9}(3x-5)^9 + \frac{125}{8}(3x-5)^8 + C\right.}$

(d) 5pts $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \frac{1}{2}\int_0^{\sqrt{\pi}} \cos(x^2)\cdot 2x\,dx = \frac{1}{2}\int_{x=0}^{x=\sqrt{\pi}} \cos(u)\,du$

$= \frac{1}{2}\left[\sin(x^2)\right]_0^{\sqrt{\pi}} = \frac{1}{2}\sin\left(\left(\sqrt{\pi}\right)^2\right) - \frac{1}{2}\sin(0)^2 = \boxed{0}$

(e) 5pts $\int e^{\sin(x)}\cos(x)\,dx = \boxed{e^{\sin(x)} + C}$

$= \int e^u\,du$, where $u = \sin(x)$ &

$du = \cos(x)\,dx$

6) $f'(x) = x^2+4x \longrightarrow$

a) 5pts Net change in f over $[-2,2]$ is $\int f'(x)\,dx = \frac{16}{3}$, by #1c

b) 5pts and Total Change in f over $[-2,2] = \int |f'(x)|\,dx = 16$ by #2c

7) a) 5pts $y = f(x) = x^2+4x$. Find f^{-1}:

$y^2+4y+2^2 = x+4$

$(y+2)^2 = x+4$

$y = -2 \pm \sqrt{x+4} \rightsquigarrow \boxed{f^{-1}(x) = \sqrt{x+4} - 2 \text{ by domain chosen}}$

b) $(f^{-1})'(x) = \frac{1}{2}(x+4)^{-\frac{1}{2}}$

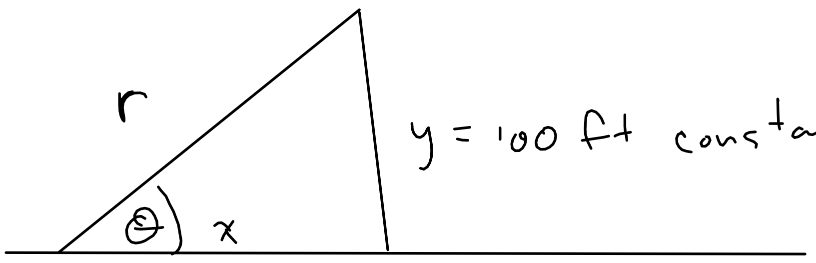
$\longrightarrow (f^{-1})'(12) = \frac{1}{2}(12+4)^{-\frac{1}{2}} = \frac{1}{2}(16)^{-\frac{1}{2}} = \frac{1}{2}\left(\frac{1}{4}\right) = \boxed{\frac{1}{8} = (f^{-1})'(12)}$

c) 5pts $(f^{-1})(12) = \sqrt{12+4} - 2 = \sqrt{16} - 2 = 4-2 = 2 = f^{-1}(12)$

$(f^{-1})'(12) = \frac{1}{f'(f^{-1}(12))} = \frac{1}{f'(2)} = \frac{1}{2(2)+4} = \boxed{\frac{1}{4+4} = \frac{1}{8} = (f^{-1})'(12)}$

$f'(x) = 2x+4$

(8)



$\frac{dx}{dt} = 8 \frac{ft}{sec}$

Let θ = angle string makes with the horizontal
x = horizontal distance the kite is from person
r = amt of string that's been let out.

Want to know $\left.\frac{d\theta}{dt}\right|_{r=200 \text{ ft.}}$

$\tan\theta = \frac{100}{x} = 100x^{-1}$

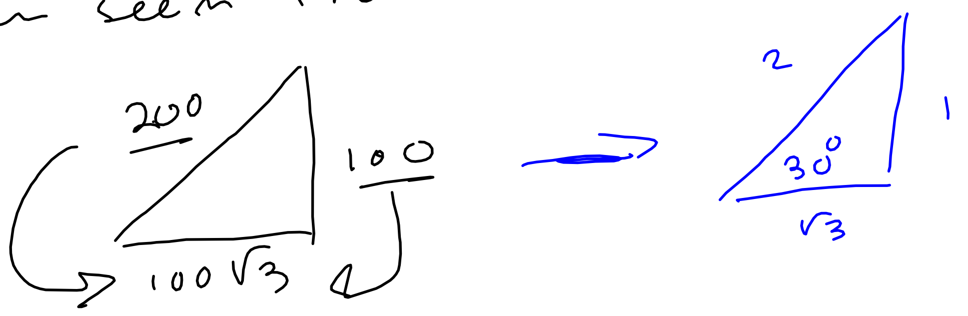
$\rightarrow \boxed{\sec^2\theta \cdot \frac{d\theta}{dt} = -100x^{-2} \cdot \frac{dx}{dt}}$ Save

$r = 200$ and $x^2 + 100^2 = r^2$ $\quad (2 \cdot 100)^2 = 2^2 \cdot 100^2 = 4 \cdot 100^2$

$\rightarrow x^2 = r^2 - 100^2 = 200^2 - 100^2 = 4(100^2) - 100^2 = 3 \cdot 100^2$

$\rightarrow x = \sqrt{3 \cdot 100^2} = 100\sqrt{3}$

This is evidently a $1-2-\sqrt{3}$ right triangle, which should have been seen from



Recall

$\sec^2\theta \cdot \frac{d\theta}{dt} = -100x^{-2} \cdot \frac{dx}{dt} = -\frac{100}{x^2}\frac{dx}{dt}$

$\rightarrow \sec^2(30°)\frac{d\theta}{dt} = -\frac{100}{(100\sqrt{3})^2} \cdot 8$

$\rightarrow \left(\frac{2}{\sqrt{3}}\right)^2 \frac{d\theta}{dt} = \frac{100}{100^2} \cdot \frac{1}{\sqrt{3}} \cdot \frac{dx}{dt}$

Still need $\frac{dy}{dt}$: $x^2 + 100^2 = r^2 \rightarrow$

$2x\frac{dx}{dt} = 2r\frac{dr}{dt} \rightarrow$

$100\sqrt{3}\,\frac{dx}{dt} = 200 \cdot 8 \frac{ft}{s}$

$\rightarrow \frac{dx}{dt} = \frac{200 \cdot 8}{100\sqrt{3}} = \frac{16}{\sqrt{3}}$ OR $\frac{16\sqrt{3}}{3} = \frac{dx}{dt}$

Now, combining with above

$\left(\frac{2}{\sqrt{3}}\right)^2 \frac{d\theta}{dt} = \frac{1}{100\sqrt{3}} \cdot \frac{16\sqrt{3}}{3} = \frac{16}{300} = \frac{8}{150} = \frac{4}{75} \rightarrow$

$\frac{d\theta}{dt} = \frac{4}{25} \cdot \frac{3}{4} = \boxed{\frac{3}{25} \frac{\text{radians}}{\text{sec}}}$

(9) (5pts) Claim: $\lim_{x \to 3} (7x+11) = 32$

Proof

Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{7}$. Then

$0 < |x-3| < \delta \Rightarrow |7x+11-32| = |7x-21| = 7|x-3| < 7\delta = \varepsilon$ ■

(10) $x^2 + 2xy + 4y^2 = 12 \Rightarrow$

(a) (5pts) $2x + 2y + 2xy' + 8yy' = 0 \Rightarrow$

$y'(2x+8y) = -2x-2y \Rightarrow$

$$y' = \frac{-2x-2y}{2x+8y}$$

(b) (5pts) $y' \Big|_{\substack{x=2 \\ y=1}} = \frac{-2(2)-2(1)}{2(2)+8(1)} = \frac{4-2}{4+8} = \frac{2}{12} = \frac{1}{6} \Rightarrow$

Eq'n of tangent line @ $(x,y) = (2,1)$ is $\boxed{y = \frac{1}{6}(x-2)+1}$

$\boxed{\text{Bonus1}}$ (5pts) Claim: $\lim_{x \to 3} (x^2-3) = 6$

Scratch $\varepsilon > 0$ given.

$$|x^2-3-6| = |x^2-9| = \underbrace{|x+3|}_{\text{Needs Bound}}\underbrace{|x-3|}_{<\delta}$$

Assume $\delta \leq 1$. Then

$x \to 3 \Rightarrow 2 < x < 4 \Rightarrow$

$2+3 < x+3 < 4+3$

$\Rightarrow |x+3| < 7$

Proof

Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{7}\}$. Then

$0 < |x-3| < \delta$ implies $|x^2-3-6| = |x^2-9| = |x+3||x-3|$

$< 7\delta \leq 7 \cdot \frac{\varepsilon}{7} = \varepsilon$ ■

Bonus 2 5 pts

$$y = \frac{(x-3)^3(x+2)^{\frac{3}{5}}}{(x+5)(x-7)^5} \longrightarrow$$

$$\ln(y) = 3\ln(x-3) + \frac{3}{5}\ln(x+2) - \ln(x+5) - 5\ln(x-7) \longrightarrow$$

$$\vdots$$

$$y' = \left(3\left(\frac{1}{x-3}\right) + \frac{3}{5}\left(\frac{1}{x+2}\right) - \frac{1}{x+5} - 5\left(\frac{1}{x-7}\right)\right)\left(\frac{(x-3)^3(x+2)^{\frac{3}{5}}}{(x+5)(x-7)^5}\right)$$

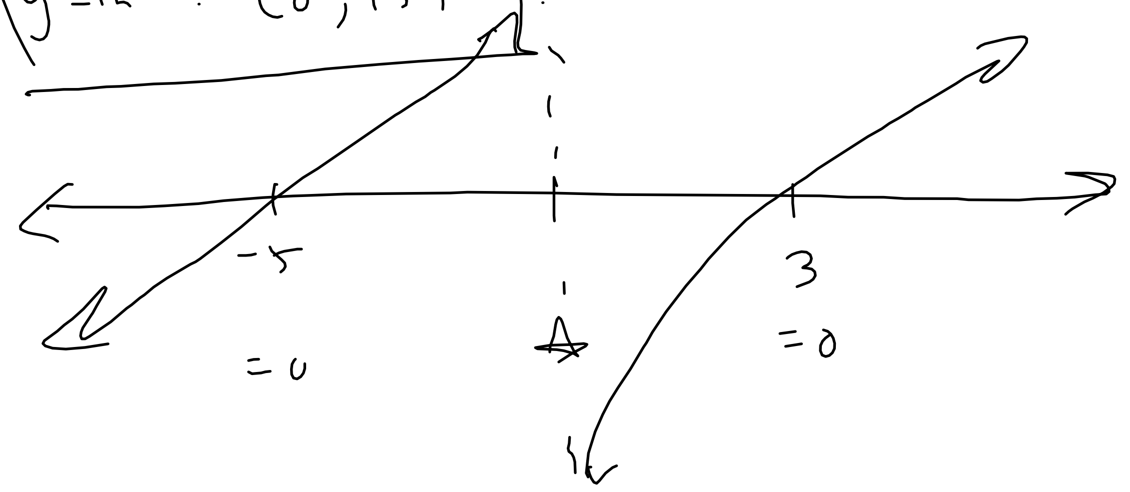
Bonus 3 $R(x) = \dfrac{(x+5)(x-3)}{x-1}$

$\boxed{\text{V.A.: } x = 1}$

$\boxed{\text{x-int: } (-5,0), (3,0)}$

$$R(0) = \frac{(5)(-3)}{-1} = 15$$

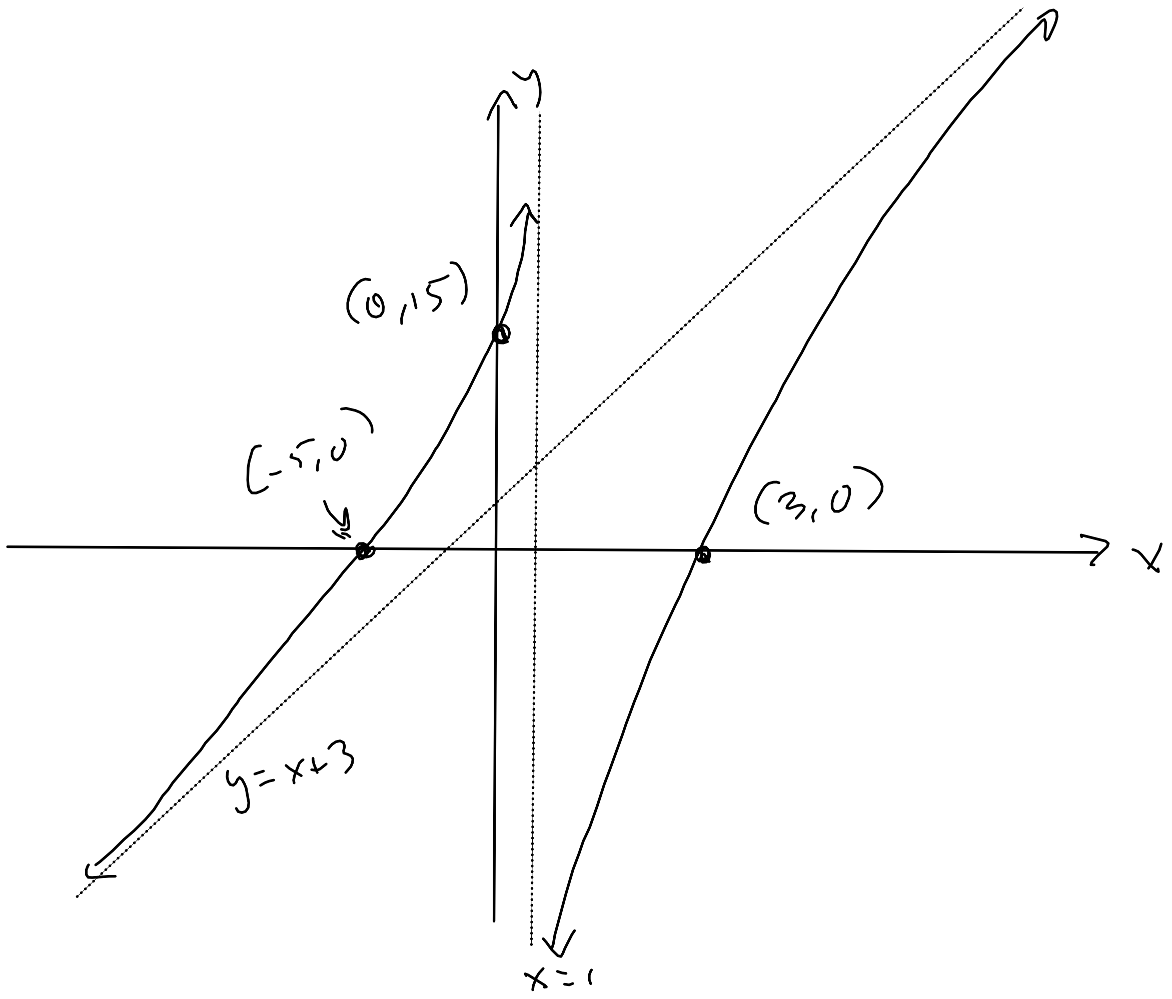
$\boxed{\text{y-int: } (0,15)}$



O.A.: $\dfrac{x^2+2x-15}{x-1}$

$$\begin{array}{r|rrr} 1 & 1 & 2 & -15 \\ & & 1 & \\ \hline & 1 & 3 & \end{array}$$

$\boxed{y = x+3 \text{ is O.A.}}$



Check the calculus:

$$R'(x) = \frac{(2x+2)(x-1) - (x^2+2x-15)(1)}{(x-1)^2}$$

$$= \frac{2x^2 - 2x + 2x - 2 - x^2 - 2x + 15}{(x-1)^2} = \frac{x^2 - 2x + 13}{(x-1)^2} \quad \text{set} = 0 \longrightarrow$$

$$x^2 - 2x = -13$$

$$x^2 - 2x + 1 = -12$$

$(x-1)^2 = -12$ No real sol'ns

$+ \quad | \quad +$ R'

1

Checks out

increasing on $(-\infty, 1) \cup (1, \infty)$

$$R''(x) = \frac{(2x-2)(x-1)^2 - (x^2-2x+13)(2(x-1))}{(x-1)^4}$$

$$= \frac{(2x-2)(x-1) - 2(x^2-2x+13)}{(x-1)^3}$$

$$= \frac{2x^2 - 4x + 2 - 2x^2 + 4x - 26}{(x-1)^3} = \frac{-24}{(x-1)^3}$$

$+ \quad | \quad -$

1

concave up: $(-\infty, -1)$

" down: $(-1, \infty)$

agrees w/ picture.