6.4 Derivatives of Logarithmic Functions.

$$\boxed{1} \quad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Recall, $\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$

We use this and $\frac{d}{dx}\left[e^x\right] = e^x$ to find $\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$

Application to integration is now we know what $\int \frac{dx}{x}$ is!

$$f(x) = e^x \implies f^{-1}(x) = \ln(x)$$

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{f'(\ln(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Book uses implicit differentiation to find $\frac{d}{dx}\left[\ln x\right]$

$y = \ln x \implies$

$y = \log_a x$ means $x = a^y$

$x = e^y$

$1 = e^y \cdot y' \implies$

$$y' = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x} = \frac{d}{dx}\left[\ln(x)\right]$$

2 $$\boxed{\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}} \quad \text{or} \quad \boxed{\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}}$$

$$\frac{d}{dx}\left[\ln(u)\right] = \frac{d}{du}\left[\ln u\right] \cdot \frac{du}{dx} = \frac{1}{u} \cdot u' = \frac{u'}{u}$$

$$\frac{d}{dx}\left[\ln(\sin x)\right] = \frac{\cos x}{\sin x} = \cot x$$

3 $$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}}$$

4 $$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

Chain Rule Versions

$$\frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

u $\frac{df}{du}$ $\frac{du}{dx}$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \; !$$

$\frac{d}{dx}[\ln|x|]$:

$x > 0$ $\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln x] = \frac{1}{x}$

$x < 0$ $\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln(-x)] = \frac{-1}{-x} = \frac{1}{x}$

$u(x) \Rightarrow \frac{u'}{x} = -1$

5 $\boxed{\int \tan x\,dx = \ln|\sec x| + C}$

$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$!

$\int \tan x\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\int \frac{f'(x)}{f(x)}\,dx = -\int \frac{du}{u}$

$f(x) = \cos x$

$f'(x) = -\sin x \Longrightarrow$

$u = f(x)$

$\frac{du}{dx} = f'(x)$

$du = f'(x)\,dx$

$= -\ln|f(x)| + C$

$= -\ln|\cos x| + C$

$= \ln\left(|\cos x|^{-1}\right) + C$

$= \ln\left(\frac{1}{|\cos x|}\right) = \ln\left(\left|\frac{1}{\cos x}\right|\right) + C$

$= \ln|\sec x| + C$

6

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Change of Base!

$$\frac{d}{dx}\left[\log_3 x\right] = \frac{d}{dx}\left[\frac{\ln x}{\ln 3}\right] = \frac{1}{\ln 3}\frac{d}{dx}\left[\ln x\right] = \frac{1}{\ln 3}\cdot\frac{1}{x} = \frac{1}{x \ln a}$$

$$= \frac{1}{(\ln a)\, x}$$

7

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$(a^b)^c = a^{bc}$$

$$a^{bc} = (a^b)^c$$

$$a^x = e^{\ln(a^x)} = e^{x\ln(a)} = e^{(\ln(a))x}$$

$$\frac{d}{dx}[a^x] = \frac{d}{dx}\left[e^{(\ln a)x}\right] = (\ln a)e^{(\ln a)x} = (\ln a)\left(e^{(\ln a)}\right)^x$$

$$\boxed{(\ln a)a^x = \frac{d}{dx}[a^x]}$$

$$e^{5x} \longrightarrow 5e^{5x}$$

Logarithmic Differentiation

Differentiate $y = \dfrac{x^{5/3}\sqrt[4]{x^2-2x}}{(2x-7)^3}$

Recall

$$\frac{d}{dx}\left[\ln(f(x))\right] = \frac{f'(x)}{f(x)}$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$$

Really Need this for stuff like x^{x^2-5}

Base is variable & Power is variable

$$y = \frac{\sqrt[4]{x^2-2x}}{(2x-7)^3} \Rightarrow$$

$\ln(x(x-2))$

$$\ln y = \ln(\text{mess}) = \frac{1}{4}\ln(x^2-2x) - 3\ln(2x-7)$$

$\frac{1}{4}\ln x + \frac{1}{4}\ln(x-2)$

$$\Rightarrow \frac{d}{dx}\left[\text{Both Sides}\right]$$

$$\frac{y'}{y} = \frac{1}{4}\left(\frac{2x-2}{x^2-2x}\right) + 3\left(\frac{2}{2x-7}\right) \Rightarrow$$

$$y' = \left(\frac{1}{4}\left(\frac{2x-2}{x^2-2x}\right) + \frac{6}{2x-7}\right)\underbrace{\left(\frac{\sqrt[4]{x^2-2x}}{(2x-7)^3}\right)}_{y}$$

$$y = x^{x^2-5}$$

$$\ln(y) = \ln\left(x^{x^2-5}\right) = (x^2-5)\ln(x)$$

$$\Longrightarrow \frac{y'}{y} = 2x\ln x + (x^2-5)\left(\frac{1}{x}\right)$$

$$y' = \left(2x\ln x + \frac{x^2-5}{x}\right)x^{x^2-5}$$

8 $$\boxed{e = \lim_{x \to 0} (1 + x)^{1/x}}$$

Equivalently

9 $$\boxed{e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}}$$

This version fits with Compound Interest Model, which I said I'd do, but haven't, yet.

2–26 Differentiate the function.

2. $f(x) = x \ln x - x$

3. $f(x) = \sin(\ln x)$

4. $f(x) = \ln(\sin^2 x)$

5. $f(x) = \ln \dfrac{1}{x}$

6. $y = \dfrac{1}{\ln x}$

7. $f(x) = \log_{10}(x^3 + 1)$

8. $f(x) = \log_5(xe^x)$

3 $f'(x) = \cos(\ln x) \cdot \dfrac{1}{x}$

$\dfrac{d\text{ outside}}{d\text{ inside}} \cdot \dfrac{d\text{ inside}}{d\, x}$

8 $y = \log_5(xe^x)$

$y = \dfrac{\ln(xe^x)}{\ln 5} \Rightarrow y' = \dfrac{1}{\ln 5} \cdot \left[\dfrac{1e^x + xe^x}{xe^x}\right]$

9. $f(x) = \sin x \ln(5x)$

10. $f(u) = \dfrac{u}{1 + \ln u}$

11. $G(y) = \ln \dfrac{(2y+1)^5}{\sqrt{y^2+1}}$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

13. $g(x) = \ln\left(x\sqrt{x^2 - 1}\right)$

14. $g(r) = r^2 \ln(2r + 1)$

15. $f(u) = \dfrac{\ln u}{1 + \ln(2u)}$

16. $y = \ln|1 + t - t^3|$

17. $f(x) = x^5 + 5^x$

18. $g(x) = x \sin(2^x)$

Assume it's $\dfrac{df}{du}$

u is independent variable.

19. $y = \tan[\ln(ax + b)]$

20. $H(z) = \ln \sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

21. $y = \ln(e^{-x} + xe^{-x})$

22. $y = \ln|\cos(\ln x)|$

23. $y = 2x \log_{10} \sqrt{x}$

24. $y = \log_2(e^{-x} \cos \pi x)$

25. $f(t) = 10^{\sqrt{t}}$

26. $F(t) = 3^{\cos 2t}$

27–30 Find y' and y''.

27. $y = x^2 \ln(2x)$

28. $y = \dfrac{\ln x}{x^2}$

29. $y = \ln\left(x + \sqrt{1 + x^2}\right)$

30. $y = \ln(\sec x + \tan x)$

31–34 Differentiate f and find the domain of f.

31. $f(x) = \dfrac{x}{1 - \ln(x-1)}$

32. $f(x) = \sqrt{2 + \ln x}$

33. $f(x) = \ln(x^2 - 2x)$

34. $f(x) = \ln \ln \ln x$

31 $f'(x) = \dfrac{1\,(1 - \ln(x-1)) - x\left(\frac{1}{x-1}\right)}{(1 - \ln(x-1))^2}$

$\mathcal{D}(f) = \{x \mid x-1 > 0 \text{ AND } 1 - \ln(x-1) \neq 0\}$

$= \{x \mid x > 1 \text{ and } x \neq 1+e\}$ → overlap, intersection.

$1 - \ln(x-1) = 0$

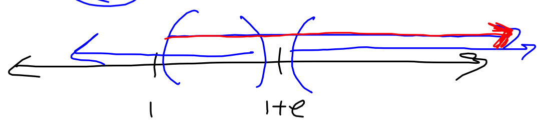
$-\ln(x-1) = -1$

$\ln(x-1) = 1$

$e^{\ln(x-1)} = e^1$

$x - 1 = e$

$x = 1 + e \notin \mathcal{D}$



$= (1, \infty) \cap [(-\infty, 1+e) \cup (1+e, \infty)]$

$= \boxed{(1, 1+e) \cup (1+e)}$

37–38 Find an equation of the tangent line to the curve at the given point.

37. $y = \ln(x^2 - 3x + 1)$, $(3, 0)$ **38.** $y = x^2 \ln x$, $(1, 0)$

$y = m_{tan}(x - x_1) + y_1$

$y = f'(x_1)(x - x_1) + f(x_1)$

$y = 1(x - 1) + 0$

$\boxed{y = x - 1}$

$f'(x) = 2x \ln x + x^2 \cdot \frac{1}{x}$

$= 2x \ln x + x$

$f'(1) = 2(1) \ln(1) + 1$

$= (2)(0) + 1$

$= 1 = f'(x_1) = m_{tan}$

40. Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

43–54 Use logarithmic differentiation to find the derivative of the function.

43. $y = (x^2 + 2)^2(x^4 + 4)^4$

44. $y = \dfrac{e^{-x}\cos^2 x}{x^2 + x + 1}$

45. $y = \sqrt{\dfrac{x - 1}{x^4 + 1}}$

46. $y = \sqrt{x}\, e^{x^2 - x}(x + 1)^{2/3}$

47. $y = x^x$

48. $y = x^{\cos x}$

49. $y = x^{\sin x}$

50. $y = \sqrt{x}^{\,x}$

51. $y = (\cos x)^x$

52. $y = (\sin x)^{\ln x}$

53. $y = (\tan x)^{1/x}$

54. $y = (\ln x)^{\cos x}$

$y = (\tan x)^{\frac{1}{x}}$

$f(x) = \frac{1}{x} = x^{-1} \Rightarrow$

$\ln y = \frac{1}{x}\ln(\tan x)$

$f'(x) = -x^{-2}$

$$\boxed{\frac{y'}{y} = -\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\frac{\sec^2 x}{\tan x}}$$

$$\frac{\sec^2 x}{\tan x} = \frac{\sec^2 x}{\left(\frac{\sin x}{\cos x}\right)} = \left(\sec^2 x\right)\left(\frac{\cos x}{\sin x}\right) = \left(\frac{1}{\cos^{2} x}\right)\left(\frac{\cancel{\cos x}}{\sin x}\right)$$

$$= \frac{1}{\cos x \sin x} = \sec x \csc x$$

$$y' = \left[-\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\frac{\sec^2 x}{\tan x}\right](\tan x)^{\frac{1}{x}} \quad \text{Good}$$

$$= \left[-\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\sec x \csc x\right](\tan x)^{\frac{1}{x}}$$

55. Find y' if $y = \ln(x^2 + y^2)$.

$$y' = \frac{2x + 2yy'}{x^2+y^2}$$

$$y'(x^2+y^2) = 2x + 2yy'$$

$$(x^2+y^2)y' - 2yy' = 2x$$

$$(x^2+y^2-2y)y' = 2x$$

$$\boxed{y' = \frac{2x}{x^2+y^2-2y}}$$

57. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

$$\frac{d}{dx}\left[y^2\right] = 2y \cdot \frac{dy}{dx}$$

59–60 Use a graph to estimate the roots of the equation correct to one decimal place. Then use these estimates as the initial approximations in Newton's method to find the roots correct to six decimal places.

59. $(x-4)^2 = \ln x$

60. $\ln(4-x^2) = x$

$f(x) = x - \ln(4-x^2) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

See 6.3 Spreadsheet, if you're interested in Newton's method for this.

Recursion methods are good to know, even though you don't have to know them, since they're built into the tech.

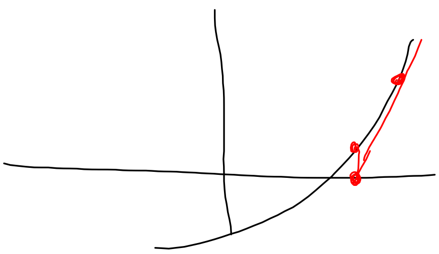
Tan line $\overset{SET}{=} 0$ for next x_{n+1}:

$y = f'(x_n)(x-x_n) + f(x_n) \overset{SET}{=} 0$

$f'(x_n)(x-x_n) = -f(x_n)$

$x - x_n = -\dfrac{f(x_n)}{f'(x_n)}$

$x = x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$



63–66 Discuss the curve under the guidelines of Section 3.5.

63. $y = \ln(\sin x)$

64. $y = \ln(\tan^2 x)$

65. $y = \ln(1 + x^2)$

66. $y = \ln(x^2 - 3x + 2)$