1. (10 pts) Use the limit definition of the definite integral to evaluate $\int_0^3 (x^2 - 2x)\,dx$.

$\Delta x = \frac{3-0}{n} = \frac{3}{n}$, $x_k = 0 + k\Delta x = \frac{3k}{n}$ $\quad \int_0^3$

$\sum_{k=1}^{n} f(x_k)\Delta x = \Delta x \sum_{k=1}^{n} f(x_k) = \frac{3}{n}\sum_{k=1}^{n}(x_k^2 - 2x_k) = \frac{3}{n}\sum_{k=1}^{n}\left(\left(\frac{3k}{n}\right)^2 - 2\left(\frac{3k}{n}\right)\right)$

$= \frac{3}{n}\sum \frac{9k^2}{n^2} - \frac{3}{n}\sum \frac{6k}{n} = \frac{3}{n}\cdot\frac{9}{n^2}\sum_{k=1}^{n} k^2 - \frac{18}{n^2}\sum_{k=1}^{n} k$

$= \frac{27}{\cancel{n^3}}\cdot\frac{\cancel{n^3}+\ldots}{3} - \frac{18}{\cancel{n^2}}\cdot\frac{\cancel{n^2}+\ldots}{2} \xrightarrow{n\to\infty} 9 - 9 = \boxed{0.}$

2. (15 pts) Use the Fundamental Theorem of Calculus (Part II) to evaluate $\int_0^3 (x^2 - 2x)\,dx$

$= \left[\frac{x^3}{3} - x^2\right]_0^3 = \frac{3^3}{3} - 3^2 - (0 - 0) = 9 - 9 = \boxed{0}$ ✓

B1 (Bonus 5 pts) What would the x_k be if the interval were [1, 3] instead of [0, 3], in Problem #1? I just want to see the x_k. Don't do any more than that.

$\rightarrow [a, b] = [1, 3] \quad a \neq 0$

$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$

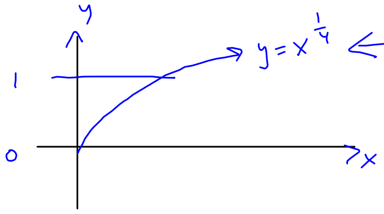
$x_k = a + k\Delta x = 1 + k\left(\frac{2}{n}\right) = \boxed{1 + \frac{2k}{n}}$

3. $20\ pts$ ~~(10 pts)~~ Show that $\int_0^1 y^4\,dy = \int_0^1 (1-x^{1/4})\,dx$ by evaluating each, separately. $\int_0^1 y^4 = \int_0^1 (1-x^{\frac{1}{4}})$

$$\int_0^1 y^4\,dy = \frac{1}{5}y^5\Big]_0^1 = \frac{1}{5}\left[1^5 - 0^5\right] = \boxed{\frac{1}{5}}$$

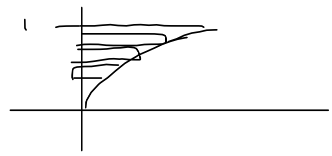
$$\int_0^1 (1-x^{\frac{1}{4}})\,dx = \left[x - \frac{4}{5}x^{\frac{5}{4}}\right]_0^1 = 1 - \frac{4}{5}(1)^{\frac{5}{4}} - (0-0) = 1 - \frac{4}{5} = \boxed{\frac{1}{5}}\,!$$

?2 (Bonus 5 pts) Draw a picture and explain what's going on in the previous problem.



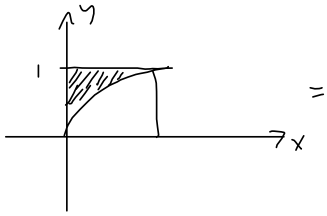
$y = x^{\frac{1}{4}} \longleftrightarrow x = y^4$

$\int_0^1 y^4\,dy$



$$\int_0^1 1\,dx \quad - \quad \int_0^1 x^{\frac{1}{4}}\,dx = \int_0^1 (1-x^{\frac{1}{4}})\,dx$$

$\int_0^1 (1-x^{\frac{1}{4}})\,dx$:



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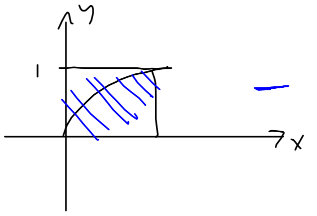

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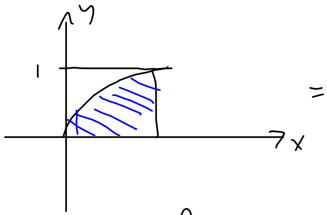


$=$

4. (10 pts) Evaluate the definite integral. $\int_0^6 |x-4|\,dx$

$$|x-4| = \begin{cases} x-4 & \text{if } x-4 \ge 0 \\ -x+4 & \text{if } x-4 < 0 \end{cases} = \begin{cases} x-4 & \text{if } x \ge 4 \quad \int_4^6 \\ -x+4 & \text{if } x < 4 \quad \int_0^4 \end{cases}$$

$$= \int_0^4 (x-4)\,dx + \int_4^6 (-x+4)\,dx \longrightarrow \text{or } -\int_4^6 (x-4)\,dx$$

$$= \left(\frac{x^2}{2} - 4x\right)_0^4 - \left(\frac{x^2}{2} - 4x\right)_4^6$$

$$= \frac{4^2}{2} - 4(4) - (0-0) - \left[\frac{6^2}{2} - 4(6) - \left(\frac{4^2}{2} - 4(4)\right)\right]$$

$$= 8 - 16 - \left[18 - 24 - (8-16)\right] = -8 - \left[-6 - (-8)\right] = -8 - [2]$$

$= -10$?! impossible!

I got the signs exactly wrong!

$$\int_0^4 (-x+4)\,dx + \int_4^6 (x-4)\,dx$$

Instead of -10, we obtain $\boxed{+10}$

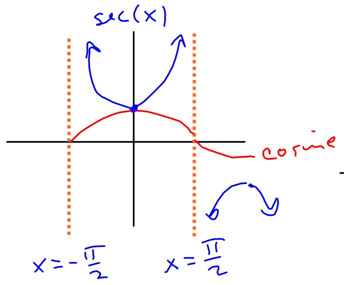
B3 (Bonus 5 pts) Evaluate the definite integral $\int_{-\pi/3}^{\pi/4} |\sec x \tan x|\, dx$

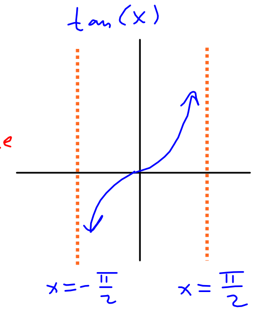
$$= -\int_{-\frac{\pi}{3}}^{0} \sec x \tan x\, dx + \int_{0}^{\frac{\pi}{4}} \sec x \tan x \, dx$$

$$= -\sec(x)\Big]_{-\frac{\pi}{3}}^{0} + \sec x \Big]_{0}^{\frac{\pi}{4}} =$$

$$= -\sec(0) - (-\sec(-\tfrac{\pi}{3})) + (\sec\tfrac{\pi}{4} - \sec(0))$$

$$= -1 + 2 + \sqrt{2} - 1 = \boxed{\sqrt{2}}$$





$\sec(x) > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$

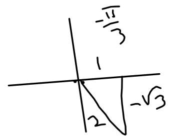
$\tan(x) < 0$ on $(-\frac{\pi}{2}, 0)$

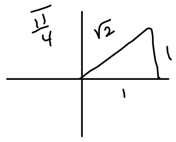
$\tan(x) \geq 0$ on $[0, \frac{\pi}{2})$

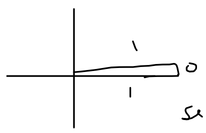
combine

$\sec x \tan x \geq 0$ on $[0, \frac{\pi}{2})$

$\&\ < 0$ on $(-\frac{\pi}{2}, 0]$







$\sec(0) = \frac{1}{\cos(0)} = 1$

5. (10 pts) Evaluate the indefinite integral. It will involve a u-substitution. $\int \frac{x}{\sqrt{2x+1}} dx$

$u = 2x+1 \Rightarrow du = 2dx \Rightarrow \frac{du}{2} = dx$

eliminate!

$$\Rightarrow \int \frac{x}{\sqrt{u}} \cdot \frac{du}{2}$$

$u = 2x+1 = u$

$2x = u-1$

$x = \frac{u-1}{2}$

$$= \frac{1}{2}\int \frac{u-1}{2} \cdot u^{-\frac{1}{2}} du$$

$$= \frac{1}{4}\int (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) du = \frac{1}{4}\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right] + C = \boxed{\frac{1}{6}u^{\frac{3}{2}} - \frac{1}{2}u^{\frac{1}{2}} + C}$$

6. ~~(10 pts)~~ 15 pts Evaluate the indefinite integral. $\int \frac{\sin x}{\cos^2 x} dx$.

2 ways: ① slick trig identity ② u-substitution.

① $\frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x$

$$\int \sec x \tan x \, dx = \boxed{\sec(x) + C}$$

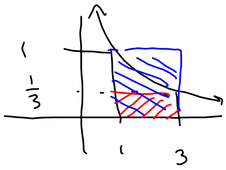
$\cos^{-1}(x)$ means $\arccos(x)$, not $\frac{1}{\cos(x)}$

② $u = \cos(x) \Rightarrow du = -\sin x \, dx$

$$\Rightarrow -\int (\cos(x))^{-2} (-\sin(x)\, dx) = -\int u^{-2} du = -\frac{u^{-1}}{-1} + C = u^{-1} + C = (\cos(x))^{-1} + C$$

$$= \frac{1}{\cos(x)} \qquad = \sec(x) + C$$

B4 (Bonus 5 pts) Find an upper bound and a lower bound for the definite integral $\int_1^3 \frac{1}{x}dx$. You do *not* know how to evaluate this integral, using only this semester's worth of training. But you *can* put a ceiling and a floor on it.



$$\int_1^3 \frac{1}{x}\,dx$$

$$m(b-a) \le \int_a^b \frac{1}{x}dx \le M(b-a)$$

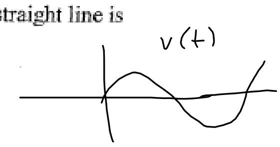
$$\frac{1}{3}(3-1) \le \int_a^b m \le 1(3-1)$$

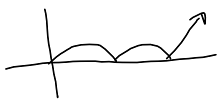
$$\boxed{\frac{2}{3} \le \int_1^3 \frac{dx}{x} \le 2}$$

7. (10 pts) Suppose the velocity (meters per second) of a particle moving in a straight line is $v(t) = -t^2 + 4t + 5$. Tell me what the two integrals represent.

a. $\int_0^6 v(t)dt$ Net Change

b. $\int_0^6 |v(t)|dt$



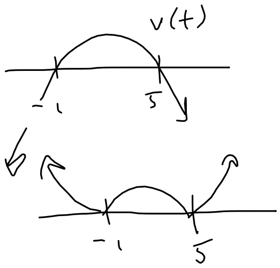


a) Net Change in position

b) Total distance covered

35 (Bonus 5 pts) Evaluate the 2nd integral in the previous problem: $\int_0^6 |v(t)|dt$

$v(t) = -t^2+4t+5 = -(t^2-4t-5) = -(t-5)(t+1)$



$$|v(t)| = \begin{cases} v(t) & \text{if } v(t) \ge 0 \\ -v(t) & \text{if } v(t) < 0 \end{cases}$$

$t \in [0,6]$

$$\int_0^6 |v(t)|\,dt = \int_0^5 v(t)\,dt - \int_5^6 v(t)\,dt$$

$$= \int_0^5 (-t^2+4t+5)\,dt - \int_5^6 (-t^2+4t+5)\,dt$$

3 36
6
216.

$$= \left[-\frac{1}{3}t^3 + 2t^2 + 5t\right]_0^5 - \left[-\frac{1}{3}t^3 + 2t^2 + 5t\right]_5^6$$

$$= -\frac{1}{3}(5)^3 + 2(5)^2 + 5(5) - (0+0+0) - \left[-\frac{1}{3}(6)^3 + 2(6)^2 + 5(6) - \left(-\frac{1}{3}(5)^3 + 2(5)^2 + 5(5)\right)\right]$$

$$= -\frac{125}{3} + 75 - \left[-\frac{216}{3} + 72 + 30 - \left(-\frac{125}{3} + 75\right)\right]$$

$$= -\frac{125}{3} + 75 - \left[\underline{-72+72} + 30 + \frac{125}{3} - 75\right]$$

$$= -\frac{125}{3} + 75 - \left[-45 + \frac{125}{3}\right] = -\frac{125}{3} + 75 + 45 - \frac{125}{3} = 120 - \frac{250}{3}$$

$$= \frac{360-250}{3} = \boxed{\frac{110}{3}}$$

8. (10 pts) Perform the indicated differentiation: Keeps csc(x) & csc(x^3) away from 0. (continuity!)

a. $\frac{d}{dx}\int_{\pi/6}^{x} \frac{t^2-3t}{\csc^3(t)}dt$. Assume $\frac{\pi}{6} < x < \pi$.

b. $\frac{d}{dx}\int_{\pi/6}^{x^3} \frac{t^2-3t}{\csc^3(t)}dt$. Assume $\frac{\pi}{6} < x^3 < \pi$

$\int_{\frac{\pi}{6}}^{x}$

FTC I :

The derivative of the antiderivative is the integrand, loosely speaking. $\int_{\pi/6}^{x^3}$

(a) $\boxed{\frac{x^2-3x}{\csc^3(x)}}$

(b) $\left(\frac{(x^3)^2-3(x^3)}{\csc^3(x^3)}\right)(3x^2)$

Chain Rule.