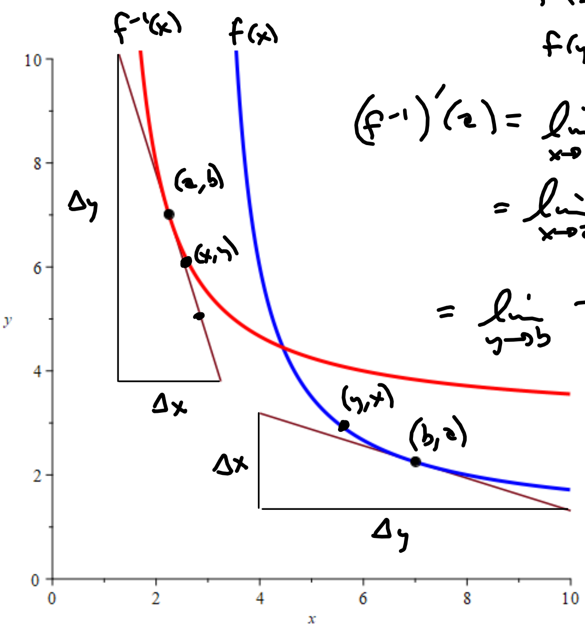
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



$$f(b) = a \implies f^{-1}(a) = b$$

$$f(y) = x \implies f^{-1}(x) = y$$

$$(f^{-1})'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}$$

$$= \lim_{x \to a} \frac{y - b}{f(y) - f(b)}$$

$$= \lim_{y \to b} \frac{1}{\frac{f(y) - f(b)}{y - b}} = \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}$$

Properties of Exponents

$(ab)^x = a^x b^x$

$a^x a^y = a^{x+y}$ $a^2 a^3 = a \cdot a \cdot a \cdot a \cdot a = a^5$

$(a^b)^c = a^{bc}$ $(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^6$

$a^{-b} = \frac{1}{a^b}$

Logs ARE the exponents.

Properties of Logs.

$y = 3^5 \longrightarrow$ $x = a^b \longrightarrow$

$\log_3(y) = 5$ $\log_a(x) = b$

$\log_b(xy) = \log_b(x) + \log_b(y)$

$(3^5)(3^7)$

$\log_3(3^5 \cdot 3^7) = \log_3(3^5) + \log_3(3^7) =$ $5+7=12$

Then take the inverse log of 12.

3^{12}

$$\log_b(a^c) = c\log_b(a)$$

$$\log_b(a^2) = \log_b(a \cdot a) = \log_b a + \log_b a = 2\log_b a$$

$$\log\left(\frac{x^5\sqrt[4]{y^3}}{z^{5/7}}\right) = 5\log x + \frac{3}{4}\log y - \frac{5}{7}\log z$$

Change of Base - The linkage between the derivative of the *natural* exponential function and all the *other* exponential functions! Basically, know everything about the *natural* base *e*, and know how to convert back and forth between *e* and every other base.

$$y = \log_b(x) \longrightarrow x = b^y$$

$$\Rightarrow \log_2(x) = \log_2(b^y) = y\log_2(b) = (\log_2(b))y$$

$$\Rightarrow y = \boxed{\frac{\log_2(x)}{\log_2(b)} = \log_b(x)} = \frac{1}{\log_2(b)} \cdot \log_2(x)$$

$$\log_3(x) = \frac{\ln(x)}{\ln(3)} \quad !$$

$$= \frac{1}{\ln(3)} \ln(x)$$

Derivatives of Exponential Functions. We discover the number *e*.

$f(x) = b^x \longrightarrow f'(x) =$

$$\lim_{h\to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h\to 0} \frac{b^x b^h - b^x}{h} = \lim_{h\to 0} \frac{1}{h}\left[b^x(b^h - 1)\right]$$

$$= b^x \lim_{h\to 0} \frac{b^h - 1}{h}$$

Numerical exploration

Find a base "e" such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$!

$\lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.693147204078$

$\lim_{h\to 0} \frac{3^h - 1}{h} \approx 1.09861234998$

Genius observation by Euler: There must be a number between 2 and 3 where this limit is 1.

Also genius: If there *is* such a "natural base," then calculus gets really slick.

Euler found this number and called it "*e*" in honor of himself.

Punchline: $e \approx 2.71828182846$

$e = 2.71828182846\ldots$

infinite, w/o repeating.

$\frac{d}{dx}\left[e^x\right] = e^x$

$\ln(x) = \log_e(x)$

e^x & $\ln(x)$ are inverse functions.

$3^x = \left(e^{\ln(3)}\right)^x$

$= e^{(\ln(3))x}$

Chain rule:

$\frac{d}{dx}\left[e^x\right] = e^x \longrightarrow$

$\frac{d}{dx}\left[e^{f(x)}\right] = f'(x)\, e^{f(x)}$

$\frac{d}{dx}\left[e^{(\ln(3))x}\right] = \frac{d}{dx}\left[e^{f(x)}\right] = (\ln(3))\, e^{(\ln(3))x}$

$= (\ln(3))\, 3^x$

where $f(x) = (\ln(3))x$

$\longrightarrow f'(x) = \ln(3)$

$\frac{d}{dx}\left[3^x\right] = \ln(3)\cdot 3^x$

$$\boxed{\frac{d}{dx}\left[b^x\right] = (\ln(b))\, b^x}$$

$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$!

$f(x) = e^x$

$f^{-1}(x) = \ln(x)$

$$\boxed{\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}}\ !$$

Chain Rule: $\boxed{\frac{d}{dx}\left[\ln(f(x))\right] = \frac{1}{f(x)}\cdot f'(x) = \frac{f'(x)}{f(x)}}$

Old Finals are a good study guide.

Click here for old finals.

Today, we covered the theory of Sections 6.2 - 6.4, at least for practical purposes of differentiating exponential and logarithmic functions. We didn't mention things like:

If f is cnt^{s}, then so is f-1.

If f is difb^{l}, then so is f-1.

If f is increasing (decreasing), then so f-1 is increasing (decreasing).