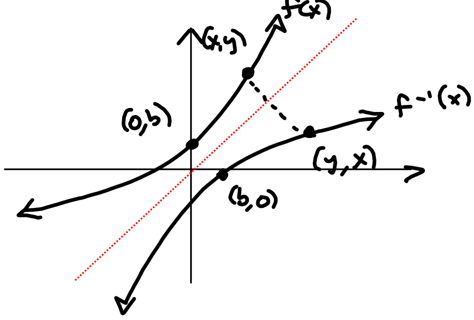
A *function* is a rule that assigns to each *x* in one set, called the domain to *exactly one y* in another set, called the *range*.

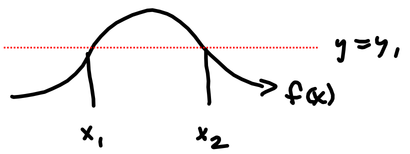
Vertical line test.

Given a function f, the function f^{-1} = "f inverse" is ~~a~~ the function satisfying $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

If (x,y) is on the graph of $f(x)$, then (y,x) is on the graph of $f^{-1}(x)$. This is a reflection about the line $y=x$



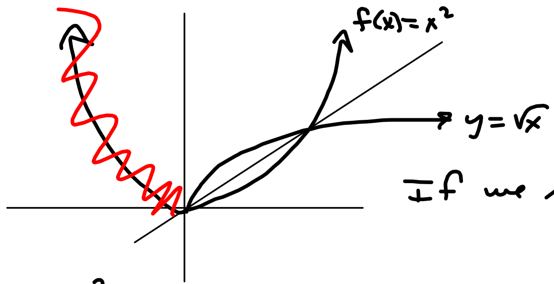
EVERY function has an inverse *relation*. Not all inverse relations are functions. For the inverse relation to be a *function, f* must be 1-to-1, i.e., each *y* in the range corresponds to exactly one *x* in the domain.



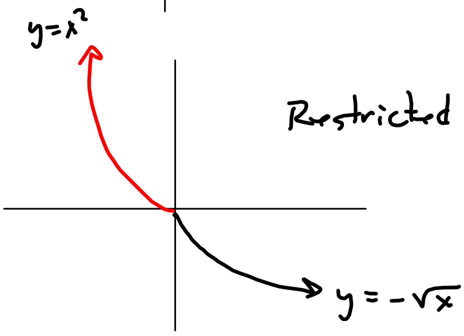
$y_1 = f(x_1) = f(x_2)$ and $x_1 \neq x_2$, i.e., not 1-to-1.

For special functions whose inverses we really want, we restrict the domain to keep it 1-to-1.

$f(x) = x^2$ (Restrict to $[0,\infty) = D$)

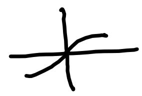


If we restrict Domain to $[0,\infty)$



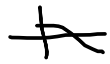
Restricted $D = (-\infty, 0]$

sine - $x \in [-\frac{\pi}{2}, \frac{\pi}{2}] = D$, $R = [-1, 1]$

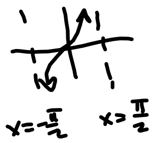


arcsine = "$\sin^{-1}$", $[-1,1] = D$, $R = [-\frac{\pi}{2}, \frac{\pi}{2}]$

cosine : $[0, \pi]$



tangent : $(-\frac{\pi}{2}, \frac{\pi}{2})$



Let $f(x) = x^2 - 4x + 5$

Give a restriction on its domain that makes it 1-to-1.

$x^2 - 4x + 5 = x^2 - 4x + 2^2 - 4 + 5 = (x-2)^2 + 1$

I suggest $D = [2, \infty)$. Since $R = [1, \infty)$, f^{-1} will have

$D(f^{-1}) = [1, \infty)$, $R(f^{-1}) = [2, \infty)$

To find f^{-1}, swap x & y & solve for y.

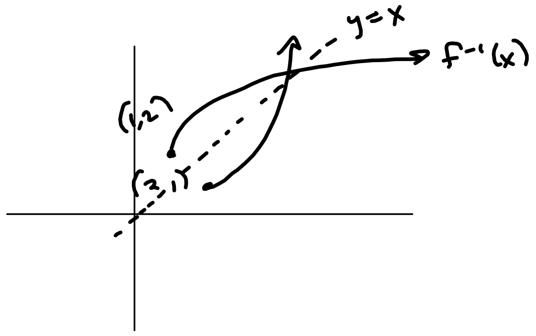
$f(x) = (x-2)^2 + 1 = y$

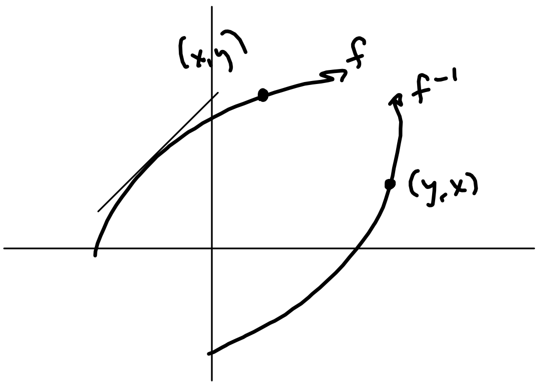
$(y-2)^2 + 1 = x$

$(y-2)^2 = x - 1$

$y - 2 = \pm\sqrt{x-1}$

$y = \pm\sqrt{x-1} + 2$, i.e., $f^{-1}(x) = 2 + \sqrt{x-1}$





$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

is a cool "cheat."

$f(x) = x^5 - 5x^3 + x - 2$

Find $(f^{-1})'(-5)$

Method 1: Find f^{-1} and calculate directly.

Method 2: Use the cheat: Solve $f(x)=5$ for x. That's

$x = f^{-1}(5)$ $\qquad x^5 - 5x^3 + x - 2 = 5 \longrightarrow$

$x = 1.$

$f'(x) = 5x^4 - 15x^2 + 1$

$f'(f^{-1}(5)) = f'(1) = 5 - 15 + 1 = -9 = (f^{-1})'(5)$

Find f^{-1} if $f(x) = \frac{x-1}{x+2}$. $D(f) = \mathbb{R} \setminus \{-2\} = \mathcal{R}(f^{-1})$.

$\frac{y-1}{y+2} = x \longrightarrow$

$\frac{x-1}{x+2} = 1 + \frac{-3}{x+2}$

$$\begin{array}{r|l} & 1 \quad r\ -3 \\ x+2 & x-1 \\ & -(x+2) \\ \hline & -3 \end{array}$$

$y-1 = x(y+2) = xy + 2x \longrightarrow$

$y - xy = 2x + 1$

$y(1-x) = 2x+1$

$y = \boxed{\frac{2x+1}{1-x} = f^{-1}(x).}$

$D(f^{-1}) = \mathbb{R} \setminus \{1\} = \mathcal{R}(f)$

Tomorrow: Derivative of the natural exponential function $y = e^x$

" " " " logarithmic function $y = \ln(x)$

I'll also give an actual argument for

$$f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$