2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

For every number $\varepsilon > 0$ there is an integer N such that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x \right| < \varepsilon$$

for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

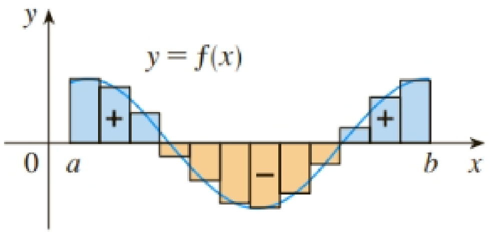


FIGURE 3

$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area.

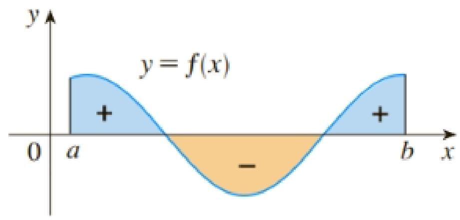


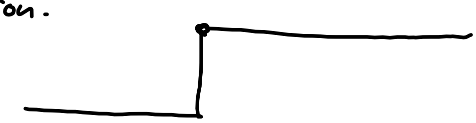
FIGURE 4

$\int_a^b f(x)\,dx$ is the net area.

3 Theorem If f is continuous on $[a, b]$, or if f has only a finite* number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

*This can be extended to "countable number" and then extended to "on a set of "zero measure," in higher mathematics.

Di-a Dirac δ- function.



4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$

Right-endpoint

8 $$\sum_{i=1}^{n} c = nc = c\sum_{i=1}^{n} 1 = c\,(\underbrace{1+1+\dots+1}_{n \text{ of 'em}})$$

9 $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

10 $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

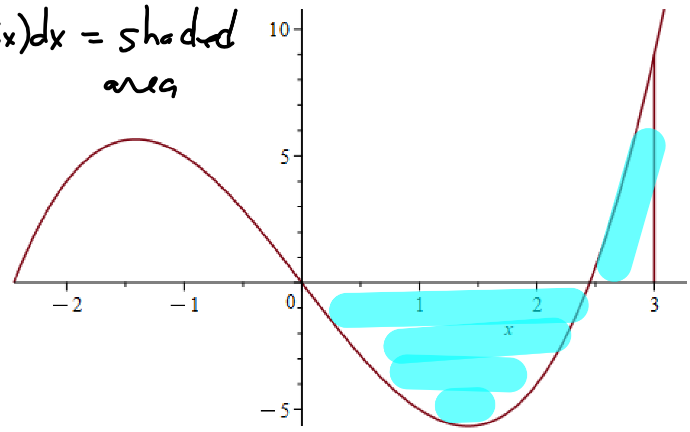
11 $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{k=1}^{n} a_k b_k = \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$$ is NOT TRUE

The sum of the products is NOT the product of the sums!

$f(x) = x^3 - 6x$

$\int_0^3 (x^3 - 6x)\,dx$ = shaded area



Signed area

"Net area"

$$\text{FTC I} - \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

$$\sum_{k=1}^{n} f(k)$$

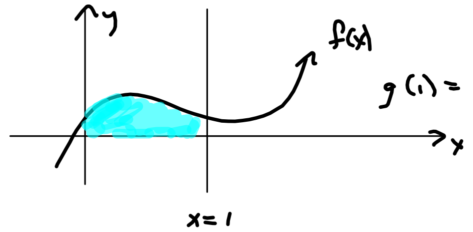
FACTS WE NEED :

$g(x) = \int_a^x f(t)\,dt$ is a function of x !

$\int_a^b f(u)\,du$ = Real # if f is cnt^{s}

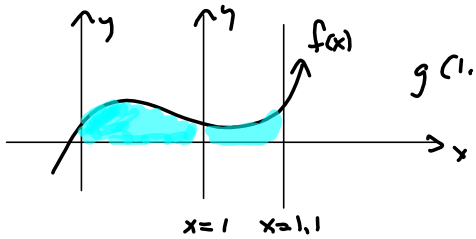
$$\int_0^1 f(x)\,dx \quad \text{-vs-} \quad \int_0^{1.1} f(t)\,dt = \int_0^{1.1} f(x)\,dx$$

$g(1)$



$$g(1) = \int_0^1 f(x)\,dx$$

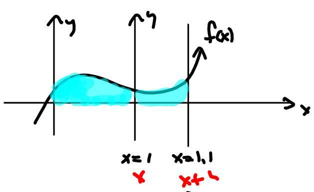
$g(1.1)$



$$g(1.1) = \int_0^{1.1} f(z)\,dz$$

$g(1)$

$$\int_0^{x+h} \quad - \quad \int_0^{x} \quad = \quad \int_x^{x+h}$$



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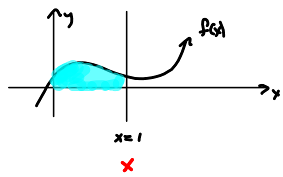

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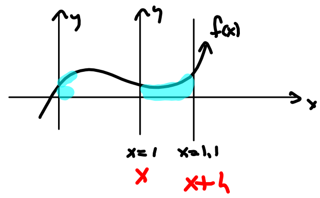


$$g(1.1) \quad - \quad g(1)$$

$$\int_0^{1.1} f(z)\,dz - \int_0^1 f(x)\,dx = \int_1^{1.1} f(u)\,du$$

$$\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

If f is cnts on $[a,b]$, then EVT says:

$\exists\, u \in [a,b] \ni f(u) \geq f(x) \ \forall\, x \in [a,b]$ $\quad \exists$ MAX

$\exists\, v \in [a,b] \ni f(v) \geq f(x) \ \forall\, x \in [a,b]$ $\quad \exists$ MIN

Extreme Value Theorem.

FTC I Let $g(x) = \int_a^x f(t)\,dt$, where $f(x)$ is cnts on $[a,b]$, $x \in [a,b]$. Then $g'(x) = f(x)$, i.e

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\left[g(x+h)-g(x)\right] = \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt\right]$$

$= \frac{1}{h}\left[\int_x^{x+h} f(t)\,dt\right]$ We want to take $h \to 0$!

Consider $\int_x^{x+h} f(t)\,dt$

$x, x+h \in [a,b]$. By EVT $\exists\, u, v \in [x, x+h]$ such that

$f(v) \leq f(z) \leq f(u) \quad \forall\, z \in [x, x+h]$

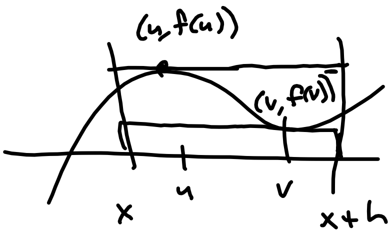
Assume $h > 0$. Then

$$h f(v) \leq \int_x^{x+h} f(t)\,dt \leq h f(u)$$

$$f(v) = \frac{h f(v)}{h} \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq \frac{h f(u)}{h} = f(u)$$

$h \to 0$

$$f(x) \leq f(x) \leq f(x)$$



By Squeeze Theorem

$f(x) = g'(x)$ ∎

Differentiate wrt x

$$\int_0^x \frac{t^3-5\cos t}{\sin^4(t^2)+1}\,dt$$

$$\frac{d}{dx}\left[\int_0^x \frac{t^3-5\cos t}{\sin^4(t^2)+1}\,dt\right] = \frac{x^3-5\cos(x)}{\sin^4(x^2)+1}$$

Chain Rule :

$$\frac{d}{dx}\left[g(u)\right] = g'(u)\,u'(x) = \frac{dg}{du}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[\int_0^x \frac{t^3-5\cos t}{\sin^4(t^2)+1}\,dt\right] = \frac{x^3-5\cos(x)}{\sin^4(x^2)+1}$$

$$\frac{dg}{dx}$$

$$\frac{d}{dx}\left[\int_0^{\tan(x)} \frac{t^3-5\cos t}{\sin^4(t^2)+1}\,dt\right] = \left(\frac{\tan^3(x)-5\cos(\tan(x))}{\sin^4(\tan^2(x))+1}\right)\left(\sec^2(x)\right)$$

$$\frac{dg(\tan(x))}{d(\tan(x))}\cdot\frac{d(\tan(x))}{dx}$$

$$\frac{dg}{du}\cdot\frac{du}{dx}$$

FTC II - 2nd Fundamental Theorem of Calculus

If f is cnt^{s} on $[a,b]$ and F is any antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$

$$\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_{x=0}^{x=1} = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}\left[1^3 - 0^3\right] = \frac{1}{3}$$

$\Delta x = \frac{1}{n}$

$x_k = \frac{k}{n}$

$$\Delta x \sum f(x_k) = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2 = \frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2} = \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\left[\frac{n^3 + \dots}{3}\right] \xrightarrow{n \to \infty} \frac{1}{3}$$

$$\frac{1}{n^3}\left[\frac{n^3}{3} + an^2 + bn + c\right]$$

Pf

Define $g(x) = \int_a^x f(t)\,dt$. Then $g'(x) = f(x)$, so $g(x)$ is an antiderivative of $f(x)$. $\therefore$ $g(x) = F(x) + C$ for some $C \in \mathbb{R}$.

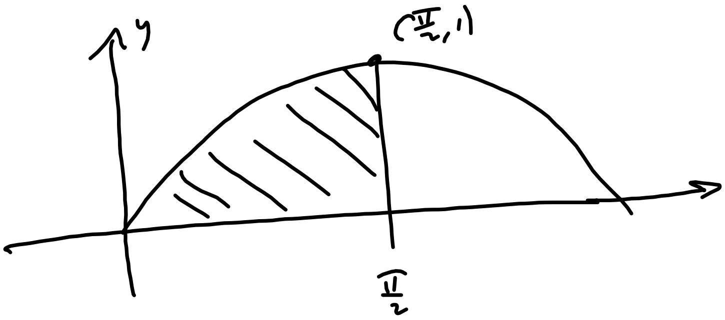
Then $g(a) = 0$ and $g(b) = \int_a^b f(t)\,dt$

Then $g(b) - g(a) = F(b) + C - (F(b) + C)$

$$= F(b) - F(a) = \int_a^b f(x)\,dx \quad \blacksquare$$

$$\int_0^{\frac{\pi}{2}} \sin(x)\,dx = \Big[-\cos(x)\Big]_0^{\frac{\pi}{2}} = -\cos\left(\tfrac{\pi}{2}\right) - \left(-\cos(0)\right)$$

$= 0 + 1 = 1$ = area under sine from 0 to $\frac{\pi}{2}$



$x_1 \quad a + \frac{\Delta x}{2}$ $\qquad 2k-1$

$x_2 = x_1 + \Delta x = a + \frac{\Delta x}{2} + \Delta x = a + \frac{3\Delta x}{2} \qquad 2k-1$

$\vdots$

$x_k = a + \frac{(2k-1)}{2} \Delta x$

$[a,b] = [3,7]$

$n = 4$ rectangles

$\Delta x = \frac{7-3}{4} = \frac{4}{4} = 1$

$x_1 = 3 + \frac{1}{2} = \frac{7}{2}$

$x_2 = 1 + \frac{7}{2} = \frac{9}{2}$

$x_3 = \frac{9}{2} + 1 = \frac{11}{2}$

$f(x) = x^2$

$x_4 = \frac{13}{2}$

$S_4 = [f(x_1) + f(x_2) + f(x_3) + f(x_4)] \Delta x$

$= \left[\left(\frac{7}{2}\right)^2 + \left(\frac{9}{2}\right)^2 + \left(\frac{11}{2}\right)^2 + \left(\frac{13}{2}\right)^2\right] (1)$