Right-Endpoint

$x_k = a + k\left(\frac{b-a}{n}\right)$, $k = 1, 2, \dots, n$

Midpoint

$x_1 = a + \frac{1}{2}\left(\frac{b-a}{n}\right)$

$x_2 = a + \frac{1}{2}\left(\frac{b-a}{n}\right) + \frac{b-a}{n}$

$= \frac{b-a+2b-2a}{2n} + a$

$= a + \frac{3b-3a}{2n} = a + \frac{3}{2}\left(\frac{b-a}{n}\right)$

$x_3 = a + \frac{3}{2}\left(\frac{b-a}{n}\right) + \left(\frac{b-a}{n}\right)\left(\frac{2}{2}\right) = \frac{5}{2}\left(\frac{b-a}{n}\right) = \frac{2(3)-1}{2}\left(\frac{b-a}{n}\right) = \frac{2k-1}{2}\left(\frac{b-a}{n}\right)$

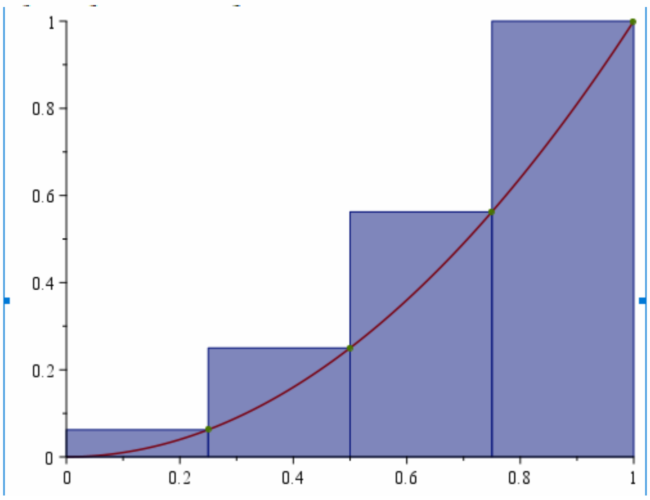
$\vdots$

$x_k = a + \left(\frac{2k-1}{2}\right)\left(\frac{b-a}{n}\right) = \frac{(2k-1)(b-a)}{2n}$, $k = 1, 2, \dots, n$

Left:

$x_k = a + (k-1)\left(\frac{b-a}{n}\right)$, $k = 1, 2, \dots, n$

An animated right Riemann sum approximation of $\int_0^1 f(x)\,dx$, where $f(x) = x^2$ and the partition is uniform. The approximate value of the integral is 0.4687500000 . Number of subintervals used: 4.

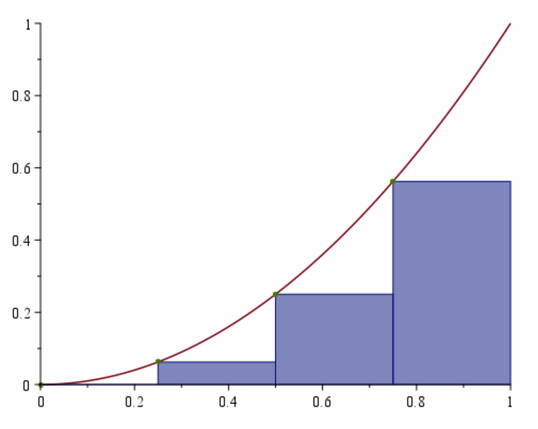


We approximate the area under y = x^2.

We're using 4 rectangles of equal width, and Right Endpoints.

Notice this is an *Upper Estimate*

Right Endpoints and Increasing is what made this happen.



Left Endpoints of an Increasing Function

gives a *Lower Estimate* of the actual area.

This is because (again) y = x^2 is increasing.

The *actual* area is greater or equal to all the lower estimates and less than or equal to all the upper estimates.

An animated left Riemann sum approximation of $\int_0^1 f(x)\,dx$, where $f(x) = x^2$ and the partition is uniform. The approximate value of the integral is 0.2187500000 . Number of subintervals used: 4.

Right Endpoint Riemann Sum

$n = 4$ by hand.

width of each rectangle. $\frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} = \frac{1}{4}$ (Interval is $[a,b] = [0,1]$

All equal widths. As $n \to \infty$, width $\to 0$.

$x_0 = a$, $x_1 = a + \text{width} = a + \frac{b-a}{n} = 0 + \frac{1}{4} = \frac{1}{4}$

$x_2 = x_1 + \text{width} = x_1 + \frac{b-a}{n} = x_0 + \frac{b-a}{n} + \frac{b-a}{n} = a + 2\left(\frac{b-a}{n}\right)$

$= 0 + 2\left(\frac{1}{4}\right)$

$x_3 = 3\left(\frac{1}{4}\right)$

$\vdots$

$x_k = k\left(\frac{1}{4}\right) = \frac{k}{4}$, $k = 1, 2, 3, 4$

$f(x_k) = x_k^2 = \left(\frac{k}{4}\right)^2 = \frac{k^2}{16} = \frac{k^2}{n^2}$

$$Area = A_1 + A_2 + A_3 + A_4 = \sum_{k=1}^{4} A_k$$

$$= f(x_1)\left(\frac{b-a}{n}\right) + f(x_2)\left(\frac{b-a}{n}\right) + f(x_3)\left(\frac{b-a}{n}\right) + f(x_4)\left(\frac{b-a}{n}\right)$$

$$= \left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) + \left(1^2\right)\left(\frac{1}{4}\right)$$

$$= \left(\frac{1}{4}\right)\left(\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + (1)^2\right)$$

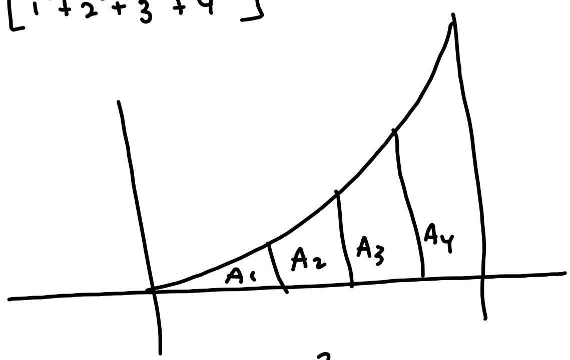
$$= \frac{1}{4}\left(\frac{1}{4^2}\right)\left(1^2 + 2^2 + 3^2 + 4^2\right)$$

$$= \frac{1}{4^3}\left(\frac{4(5)(9)}{6}\right) = \frac{1}{4^3(6)}\left((4)(5)(9)\right) = 0.46875$$

This is because $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$$\sum_{k=1}^{4} f(x_k)\frac{b-a}{n} = \frac{1}{4}\sum_{k=1}^{4}\left(\frac{k}{4}\right)^2 = \left(\frac{1}{4}\right)\left(\frac{1}{16}\right)\sum_{k=1}^{4} k^2$$

$$= \frac{1}{4^3}\left[1^2 + 2^2 + 3^2 + 4^2\right]$$



$$A_1 = \frac{1}{4}\left(\frac{1}{4}\right)^2$$

$$A(x_k) = \frac{1}{4}\left(f(x_k)\right) = \frac{1}{4}\left(\frac{k}{4}\right)^2$$

$n=4$ $\quad A = \frac{1}{4^3}\sum_{k=1}^{4} k^2 = \frac{1}{4^3}\left[\frac{4(4+1)(2(4)+1)}{6}\right]$

$n = \text{ANY}$ $\quad A = \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) =$

$f(x) = \frac{3}{x}$, $x=1$ to $x=2$, $n=4$

$\Delta x = \text{width} = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n} = \Delta x$

RIGHT: $x_k = a + k\Delta x = 1 + k\left(\frac{1}{n}\right) = 1 + \frac{k}{n} = \frac{n+k}{n}$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \Delta x \sum_{k=1}^{n} f(x_k) = \frac{1}{4}\sum_{k=1}^{n} \frac{3}{x_k} = \frac{1}{4}\sum_{k=1}^{n} \frac{3}{\frac{n+k}{n}} = \frac{1}{4}\sum_{k=1}^{4} \frac{3n}{n+k}$$

$$A_k = \frac{1}{n} \cdot \frac{3n}{n+k}$$

We prove $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Method: Show $\sum_{k=1}^{1} k^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{(2)(3)}{6} = 1 \overset{?}{=} \sum_{k=1}^{1} k^2 = 1^2 = 1$ ✓

Suppose this is true for some $n \geq 1$.

Then $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. We show $\sum_{k=1}^{n+1} k^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$

$n=1$ ✓

Then $\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{(n+1)^2}{1} \cdot \frac{6}{6}$$

$$= \frac{(n+1)[2n^2+n+6n+6]}{6} = \frac{(n+1)[2n^2+7n+6]}{6}$$

$$= \frac{(n+1)[(2n+3)(n+2)]}{6}$$

$2n^2+7n+6$

$$= \frac{(n+1)((n+1)+1)(2n+2+1)}{6}$$

$$= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \longrightarrow \text{It works for } n+1!$$

Gauss Proved $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ at age 6.

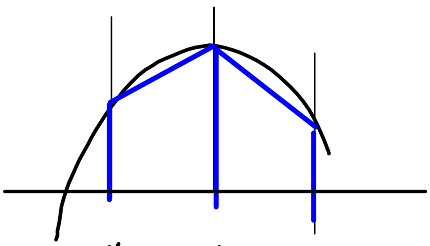
5 $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2+n}{2} = \frac{n^2}{2} + \text{lower-degree}$

$= \frac{n^2}{2} + \cdots$

6 $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+\cdots}{6} = \frac{n^3}{3} + \cdots$

7 $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^4+\cdots}{4} = \frac{n^4}{4} + \cdots$

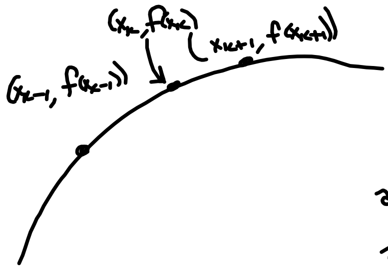
Trapezoid Rule



$$A_k = \left(\frac{b_1+b_2}{2}\right)h$$
$$= \frac{1}{2}\left(f(x_{k-1}) + f(x_k)\right)\left(\frac{b-a}{n}\right)$$
$$= \left(f(x_{k-1}) + f(x_k)\right)\left(\frac{b-a}{2n}\right)$$

$$\sum_{k=1}^{n} A_k = \frac{b-a}{2n}\left(f(x_0) + f(x_1) + f(x_1) + f(x_2) + \dots + f(x_{n-1}) + f(x_n)\right)$$
$$= \frac{b-a}{2n}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)\right)$$

Simpson's Rule



$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1)$$
$$a_1 x_2^2 + b_1 x + c_1 = f(x_2)$$
$$a_1 x_3^2 + b_1 x_3 + c_1 = f(x_3)$$

$$\left[\begin{array}{ccc|c} x_1^2 & x_1 & 1 & f(x_1) \\ x_2^2 & x_2 & 1 & f(x_2) \\ x_3^2 & x_3 & 1 & f(x_3) \end{array}\right]$$

$f(x) = x^2$ from 0 to 1

$a=0, b=1, \quad \frac{b-a}{n} = \boxed{\frac{1}{n} = \Delta x}$

$x_k = a + k\left(\frac{b-a}{n}\right) = k\left(\frac{1}{n}\right) = \boxed{\frac{k}{n} = x_k}$

$f(x_k) = x_k^2 = \left(\frac{k}{n}\right)^2 = \boxed{\frac{k^2}{n^2} = f(x_k)}$

$$A_n = \sum_{k=1}^{n} f(x_k)\Delta x = \Delta x \sum_{k=1}^{n} f(x_k) = \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{n^2} = \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\left[1^2 + 2^2 + \dots + n^2\right] = \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{1}{6n^3}\left[n(n+1)(2n+1)\right]$$

Claim:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

NOTE: $n=1$: $\frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = \frac{6}{6} = 1 = 1^2 = \sum_{k=1}^{1} k^2$

So, it works for $n=1$.

Suppose it works for $n = N$ for some $N \in \mathbb{N}$.

Then $\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$.

and so $\sum_{k=1}^{N+1} k^2 = \left(\sum_{k=1}^{N} k^2\right) + (N+1)^2$

$$= \frac{N(N+1)(2N+1)}{6} + \frac{6}{6}(N+1)^2 = \frac{N(N+1)(2N+1) + 6(N+1)^2}{6}$$

+1)

$$= \frac{(N+1)\left[N(2N+1) + 6(N+1)\right]}{6} = \frac{(N+1)\left[2N^2 + N + 6N + 6\right]}{6}$$

$$= \frac{(N+1)\left[2N^2 + 7N + 6\right]}{6} = \frac{(N+1)(2N+3)(N+2)}{6}$$