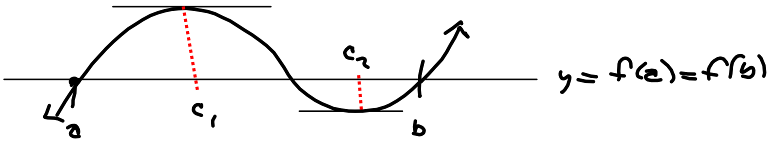
Recall:

Rolle's Theorem.

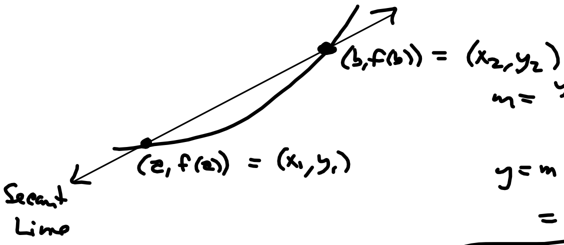
f cnt^s on $[a,b]$, difb^l on (a,b), $f(a)=f(b)$

$\longrightarrow \exists\, c \in (a,b) \ni f'(c)=0$



This one has 2. There's always at least one.

Recall: The equation of the secant line from (a, $f(a)$) to (b, $f(b)$):



$m = \dfrac{y_2-y_1}{x_2-x_1} = \dfrac{f(b)-f(a)}{b-a}$

$y = m(x-x_1)+y_1$

$= m(x-a)+f(a)$ Secant Line Eq^n

$$\boxed{y = \frac{f(b)-f(a)}{b-a}(x-a)+f(a) = g(x)}$$

Observe: $f(a)=g(a)$ and $f(b)=g(b)$.

New! We generalize Rolle's Theorem with the

MEAN-VALUE THEOREM:

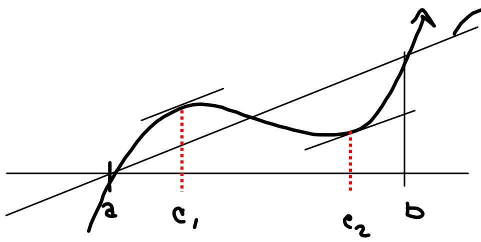
f cnt^s on $[a,b]$ f difb^l on (a,b)

$\longrightarrow$

$\exists\, c \in (a,b) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$

f cnt^s on $[a,b]$ $= \{x \mid a \le x \le b\}$ means $\lim\limits_{x\to c} f(x) = f(c)$ $\forall\, x \in [a,b]$

f difb^l on (a,b) means $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ exists $\forall\, x\in(a,b)$.



$y = \dfrac{f(b)-f(a)}{b-a}(x-a)+f(a)$

Slope of curve = Average slope over $[a,b]$ somewhere in (a,b).

Proof $h(x) = f(x) - g(x) = f(x) - \left[\dfrac{f(b)-f(a)}{b-a}(x-a)+f(a)\right]$

$h(x)$ satisfies the hypotheses of Rolle's Theorem.

$\therefore\ \exists\, c \in (a,b) \ni h'(c)=0$

$h'(c) = f'(c) - g'(x) = f'(c) - \dfrac{f(b)-f(a)}{b-a} = 0$

$\longrightarrow f'(c) = \dfrac{f(b)-f(a)}{b-a}$ ∎

Show that $f(x)= x^3+2x^2-5x+1$ satisfies the hypotheses of the Mean Value Theorem on $[1,5]$. Then find the c in the conclusion of the theorem.

Proof: f is a polynomial so f is cnt^s and dif'b^l on $(-\infty,\infty) \supset [1,5] \supset (1,5)$

we find c:

x^3+2x^2-5x+1

$$m_{sec} = \frac{f(5)-f(1)}{5-1} = \frac{151-(-1)}{4} = \frac{150}{4} = \frac{75}{2} \overset{SET}{=} f'(x):$$

$$\begin{array}{r|rrrr} 5 & 1 & 2 & -5 & 1 \\ & & 5 & 35 & 150 \\ \hline & 1 & 7 & 30 & 151 \end{array}$$

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -5 & 1 \\ & & 1 & 3 & -2 \\ \hline & 1 & 3 & -2 & -1 \end{array}$$

$f'(x) = 3x^2+4x-5 \overset{SET}{=} \frac{75}{2}$

$\Rightarrow 3x^2+4x-\frac{85}{2}$

$\frac{-5}{1}\cdot\frac{2}{2} - \frac{75}{2} = -\frac{85}{2}$

$\Rightarrow a=3,\ b=4,\ c=-\frac{85}{2}$

$\Rightarrow b^2-4ac = 4^2-\cancel{4}^{2}(3)\left(-\frac{85}{\cancel{2}}\right)$

$= 16+6(85)$

$= 16+510$

$= 526$

$\begin{array}{r} {}^{3}85 \\ \times\ 6 \\ \hline 510 \end{array}$

$2\,\overline{)526}$ $\quad 263$

$$x = \frac{-4\pm\sqrt{526}}{2(3)} = \frac{-4\pm\sqrt{526}}{6}$$

we want $\boxed{\frac{-4+\sqrt{526}}{6} = c \approx 3.15578164706 \in (1,5)}$

Proof:

Define $h(x) = f(x) - \left[\left(\frac{f(b)-f(a)}{b-a}\right)(x-a) + f(a)\right]$

$h(x)$ is dif'b'l on (a,b) & cnt's on $[a,b]$ since $f(x)$ is and $\frac{f(b)-f(a)}{b-a}(x-a) + f(a)$ is just a line.

$h(a) = f(a) - f(a) = 0$

$h(b) = f(b) - \left(\frac{f(b)-f(a)}{b-a}\right)(b-a) + f(a)$

$= f(b) - \left[(f(b)-f(a)) + f(a)\right]$

$= f(b) - f(b) + f(a) - f(a) = 0$, so

$h(x)$ satisfies Rolle's Theorem $\longrightarrow$

$\exists\, c \in (a,b) \ni h'(c) = 0$. Now

$h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \longrightarrow$

$f'(c) = \frac{f(b)-f(a)}{b-a}$ $\blacksquare$

Graphing a cubic polynomial.

$$\cancel{(x-2)(x+1) = x^2 - x - 2 = f'(x)}$$

$$\cancel{\Rightarrow f(x) = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 - 2x}$$

$$\cancel{6f(x) = 2x^3 - 3x^2 - 12x}$$

Let $f(x) = 2x^3 - 3x^2 - 12x$. Graph it.

$D = \mathbb{R} = (-\infty, \infty)$

$f(0) = 0 \rightarrow (0,0) = A$

$f(x) = 0 \Rightarrow x(2x^2 - 3x - 12) = 0$

$\Rightarrow x = 0$ OR $2x^2 - 3x - 12 = 0$

$2\left(x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2\right) = 12 + 2\left(\frac{9}{16}\right)$

$\frac{9}{8} + \frac{12}{1} \cdot \frac{8}{8} =$

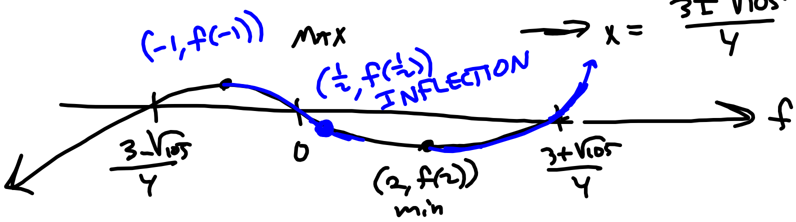
$2\left(x - \frac{3}{4}\right)^2 = \frac{105}{8}$

$= \frac{9 + 96}{8} = \frac{105}{8}$

$\left(x - \frac{3}{4}\right)^2 = \frac{105}{16}$

$\rightarrow x - \frac{3}{4} = \pm \frac{\sqrt{105}}{4}$

$\rightarrow x = \frac{3 \pm \sqrt{105}}{4} \nearrow \frac{3 + \sqrt{105}}{4}$

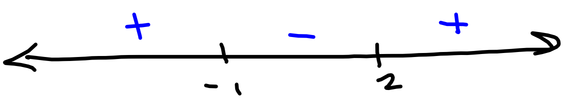


$\frac{3+10}{4} = \frac{13}{4} = 4\frac{1}{4}$

$f'(x) = x^2 - x - 2 \overset{\text{SET}}{=} 0 \Rightarrow$

$(x-2)(x+1) = 0$

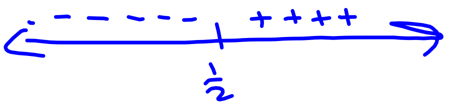
$\Rightarrow x \in \{-1, 2\}$



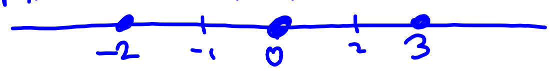
$(x+1)(x-2) = x^2 + \_\_\_$

$f'(x) = x^2 - x - 2$

→ $f''(x) = 2x - 1 \overset{\text{set}}{=} 0$ → $x = \frac{1}{2}$



Sign Pattern for $x^2 - x - 2 = (x+1)(x-2)$



$x+1$ $-2+1=-1$ "−" $0+1=1$ "+" $3+1=4$ "+"

$x-2$ $-2-2=-4$ "−" $0-2=-2$ "−" $3-2=1$ "+"