A particle moves along the curve $y = 2\sin(\pi x/2)$. As the particle passes through the point $\left(\frac{1}{3}, 1\right)$, its x-coordinate increases at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the particle to the origin changing at this instant?

A particle is moving along the curve (in cm)

$f(x) = y = 2\sin\left(\frac{\pi}{2}x\right)$, where

x & y are given in cm.

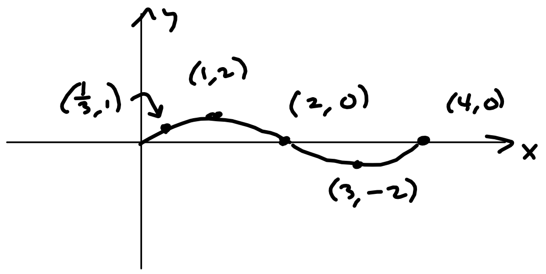
when $(x,y) = \left(\frac{1}{3}, 1\right)$, $\frac{dy}{dt} = \sqrt{10}\ \frac{cm}{sec.}$

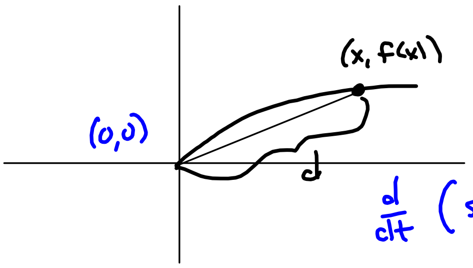
Let d = distance of the particle from the origin $(0,0)$ as a function of t = time, in seconds.

We want $\left.\frac{d}{dt}[d]\right|_{(x,y)=\left(\frac{1}{3},1\right)}$

$\frac{\pi}{2}x = 2\pi$

$x = (2\pi)\left(\frac{2}{\pi}\right) = 4$ = period





$$d = \sqrt{(x-0)^2 + \left(2\sin\left(\tfrac{\pi}{2}x\right) - 0\right)^2}$$

$$= \left(x^2 + 4\sin^2\left(\tfrac{\pi}{2}x\right)\right)^{\frac{1}{2}}$$

$$\frac{d}{dt}\left(\sin\left(\tfrac{\pi}{2}x\right)\right)^2 = 2\sin\left(\tfrac{\pi}{2}x\right)\cos\left(\tfrac{\pi}{2}x\right)\left(\tfrac{\pi}{2}\right)\frac{dx}{dx}$$

$$\longrightarrow \frac{d}{dt}[d] = \frac{1}{2}\left(x^2 + 4\sin^2\left(\tfrac{\pi}{2}x\right)\right)^{-\frac{1}{2}}\left(2x\frac{dx}{dt} + 8\sin\left(\tfrac{\pi}{2}x\right)\cdot\cos\left(\tfrac{\pi}{2}x\right)\left(\tfrac{\pi}{2}\right)\frac{dx}{dt}\right)$$

Let $(x,y) = \left(\frac{1}{3}, 1\right) \longrightarrow$

$$\left.\frac{d}{dt}\right|_{\left(\frac{1}{3},1\right)} = \frac{1}{2}\left(\left(\tfrac{1}{3}\right)^2 + 4\sin^2\left(\left(\tfrac{\pi}{2}\right)\left(\tfrac{1}{3}\right)\right)\right)^{-\frac{1}{2}}\left(2\left(\tfrac{1}{3}\right)(10) + 8\sin\left(\tfrac{\pi}{6}\right)\cos\left(\tfrac{\pi}{6}\right)\left(\tfrac{\pi}{2}\right)(10)\right)$$

$\to \sqrt{10}$ $\quad$ $\to \sqrt{10}$

$$= \frac{1}{2}\left(\frac{1}{9} + 4\left(\tfrac{1}{2}\right)^2\right)^{-\frac{1}{2}}\left(\frac{20}{3} + 8\left(\tfrac{1}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{\pi}{2}\right)(10)\right)$$

$\to \sqrt{10}$

$$= \frac{1}{2}\left(\frac{1}{9} + \frac{9}{9}\right)^{-\frac{1}{2}}\left(\frac{20}{3} + 10\sqrt{3}\,\pi\right)$$

$\to \sqrt{10}$

$$\cancel{= \frac{1}{2}\left(\frac{10}{9}\right)^{-\frac{1}{2}}\left(\frac{20}{3} + \frac{30\sqrt{3}\,\pi}{3}\right)}$$

$$\cancel{= \frac{1}{2}\left(\frac{\sqrt{9}}{\sqrt{10}}\right)\left(\frac{20 + 30\sqrt{3}\,\pi}{3}\right)}$$

$$\cancel{= \frac{1}{2\sqrt{10}}\left(20 + 30\sqrt{3}\,\pi\right)}$$

$$= \frac{1}{2}\left(\frac{10}{9}\right)^{-\frac{1}{2}}\left(\frac{20}{3} + \sqrt{30}\,\pi\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{9}}{\sqrt{10}}\right)\left(\frac{20}{3} + \sqrt{30}\,\pi\right)$$

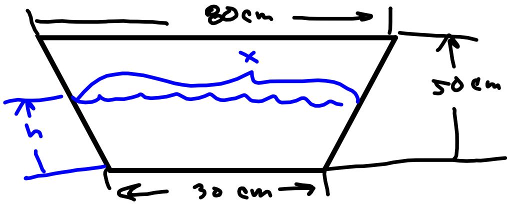
$$= \frac{1}{2}\left(\frac{3}{\sqrt{10}}\right)\left(\frac{20}{3} + \sqrt{30}\,\pi\right)$$

$$= \frac{1}{2}\left(\frac{40}{3\sqrt{10}} + \left(\frac{\sqrt{30}}{\sqrt{10}}\right)(3)\right)$$

$$= \frac{1}{2}\left(\frac{60 + 9\sqrt{30}}{3\sqrt{10}}\right)$$

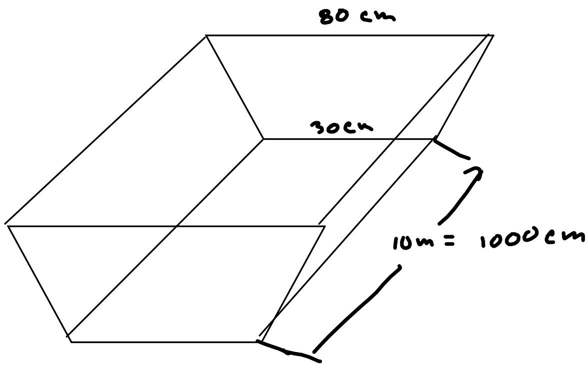
A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 30 cm deep?

h = height of water in cm.
V = volume (cm^3)

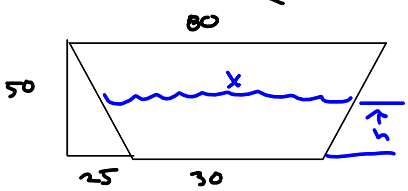


$\frac{dV}{dt} = 0.2\ m^3/min$

want $\left.\frac{dh}{dt}\right|_{h=30\,cm}$



V = Area of



x = length (width) of top of water.

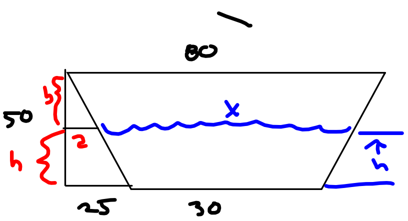
Area of trapezoid is $\frac{1}{2}(b_1+b_2)h$

$= \frac{1}{2}(30+x)$

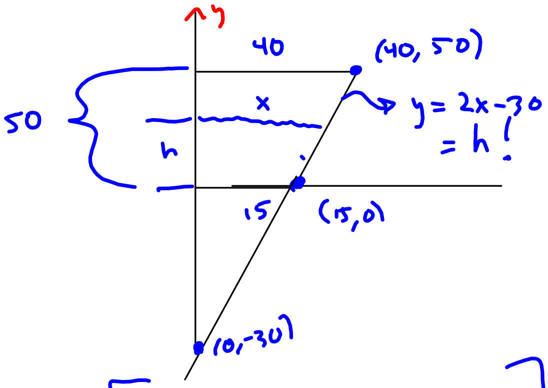
Similar triangles?

Volume of water:

$\left(\frac{1}{2}(30+x)h\right)(1000\,cm)$



$\frac{2}{1} = \frac{50}{25} = \frac{h}{a} = \frac{50-h}{a} = \frac{\frac{50}{h}}{\frac{50-h}{2}} = \frac{2}{h}$



$$\frac{50-0}{40-15} = \frac{50}{25} = 2 = \text{slope}$$

$$y = 2(x-15)+0$$

$$y = 2x-30$$

$$2x-30 = y$$

$$2x = y+30$$

$$x = \frac{y+30}{2} = \frac{h+30}{2}$$

$$\frac{1}{2}\left[\frac{1}{2}(15+x)(h)(1000)\right]$$

$$= \frac{1}{4}\left((15+x)(2x-30)(1000)\right) = \frac{1}{2} \text{ the volume}$$

$$= \frac{1}{4}\left(15+\frac{h+30}{2}\right)(h)(1000) = \frac{1}{2}V$$

$$\longrightarrow V = \frac{1}{2}\left(\frac{30+h+30}{2}\right)(h)(1000)$$

$$= \frac{1}{4}(h+60)(h)(1000)$$

$$= 250h(h+60) \longrightarrow$$

$$\frac{dV}{dt} = \left(250\frac{dh}{dt}\right)(h+60) + (250h)\left(\frac{dh}{dt}\right)$$

$$= \left(250(h+60)+250h\right)\frac{dh}{dt}$$

$$= (500h+12500)\frac{dh}{dt} \longrightarrow$$

$$\frac{dh}{dt} = \frac{\frac{dV}{dt}}{500h+250}$$

$$h = 30 \longrightarrow$$

$$\frac{dV}{dt} = 0.2\,\frac{m^3}{min} =$$

$$(1000\text{ cm})^3 = (1\text{ m})^3$$

$$10^9$$

$$\frac{dh}{dt} = \frac{10^9\text{ cm}^3/\text{min}}{500(30)+250}$$

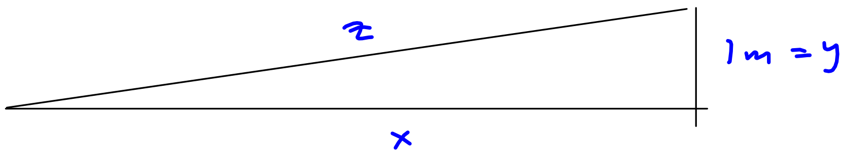
$$= \frac{10^9\text{ cm}^3/\text{min}}{15000+250\text{ cm}}$$

$$= \frac{10^9\ \frac{\text{cm}^3}{\text{min}}}{15250\text{ cm}} = \left(\frac{10^9}{15250}\right)\frac{\text{cm}^2}{\text{min}} \ ?!$$

Unit (Dimensional) analysis says we're doing this wrong.

$\frac{\text{cm}^2}{\text{min}}$? Noooooooo!

A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?



z = length of rope (m)

x = distance from dock to the boat (m)

Given $\frac{dz}{dt} = 1 \frac{m}{s}$

Want $\left.\frac{dx}{dt}\right|_{x=8}$

when $x = 8$:

$z^2 = 8^2 + 1^2 = 65$

$z = \sqrt{65}$ m

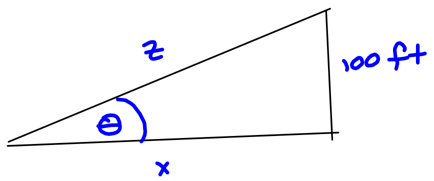
$x^2 + 1^2 = z^2$

$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$

$2(8)\frac{dx}{dt} = 2(\sqrt{65})(1)$

$$\boxed{\left.\frac{dx}{dt}\right|_{x=8} = \frac{\sqrt{65}}{8}\ \frac{m}{s}}$$

A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?



Given:

$y = 100$ ft

$\frac{dx}{dx} = 8 \frac{ft}{s}$

Lexicon: x = Horizontal distance from man to Kite

We WANT $\frac{d\theta}{dt}$ when $\left.\frac{d\theta}{dt}\right|_{z=200\text{ ft}}$

$x^2 + 100^2 = z^2$

$\frac{100}{x} = \tan\theta$

$\cot\theta = \frac{x}{100}$

$-\csc^2\theta \frac{d\theta}{dt} = \frac{1}{100}\frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{-\sin^2\theta}{100}\frac{dx}{dt}$

When $z = 200$:

$\tan\theta = \frac{100}{x}$

$\tan\theta = \frac{100}{100\sqrt{3}}$

$= \frac{1}{\sqrt{3}}$

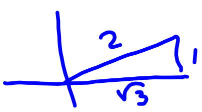
$x^2 + 100^2 = 200^2$

$x^2 = 200^2 - 100^2 = 2^2 \cdot 100^2 - 100^2$

$= 3 \cdot 100^2 = 3(10000)$

$= 30000$

$\longrightarrow x = \sqrt{3} \cdot 100 = 100\sqrt{3}$

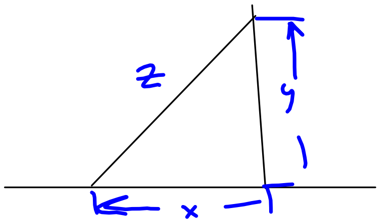


$\longrightarrow \theta = \frac{\pi}{6}$

So $\frac{d\theta}{dt} = \left(\frac{-\sin^2(\frac{\pi}{6})}{100}\right)(8)$

$= \left(\frac{(-\frac{1}{2})^2}{100}\right)(8) = \left(\frac{\frac{1}{4}}{100}\right)(8) = \frac{2}{100} = \boxed{\frac{1}{50}\ \frac{\text{radians}}{\text{sec}}}$

The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides away from the wall at a rate of 0.2 m/s. How long is the ladder?



Given $\frac{dy}{dt} = \frac{.15\,m}{s}$

Given when $x = 3\,m$, $\frac{dx}{dt} = \frac{.2\,m}{s}$

z = length of ladder (in m)

want z!

$\frac{dy}{dt} = .15\ m/sec$

$x^2 + y^2 = z^2$

$2x\,x' + 2y\,y' = 2z\,z' = 0$

$2(3)\,x' + 2(\sqrt{z^2 - x^2})(0.15) = 0$

$-$

$y^2 = z^2 - x^2$

$y = \sqrt{z^2 - x^2}$

$x^2 + y^2 = z^2$

$2(3)(.2) + 2(z^2 - 3)^{\frac{1}{2}}(.15) = 0$

$1.2 + (2\sqrt{z^2 - 3})(\frac{15}{100}) = 0$

$\frac{dy}{dt} = -0.15\ m/s.$

$\frac{3}{20}$

$\frac{3}{10}$

$=$

Off with signs

This equation will yield

$\sqrt{}$ = negative!

$1.2 - \frac{3}{10}\sqrt{z^2 - 3} = 0$

$\frac{12}{10} = \frac{6}{5}$

$\frac{3}{10}\sqrt{z^2 - 3} = \frac{6}{5}$

$\sqrt{z^2 - 3} = \frac{10}{3} \cdot \frac{6}{5} = 4$

$z^2 - 3 = 16$

$z^2 = 19$

$z = \pm\sqrt{19}$ m

$\boxed{z = \sqrt{19}\ m}$