Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.

(a) Find dF/dr and explain its meaning. What does the minus sign indicate?

(b) Suppose it is known that the earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20{,}000$ km. How fast does this force change when $r = 10{,}000$ km?

Section 2.7 - Rates of Change in Science

Marginal Cost.

Let $C(x)$ = cost in \$ of producing

x = the # of items produced.

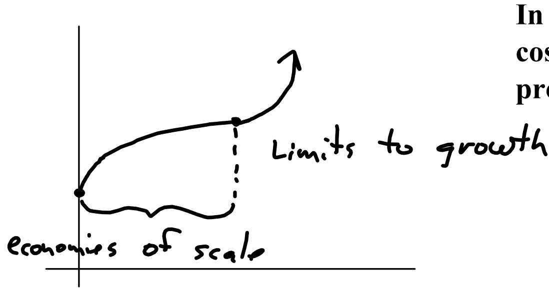
Average Rate of Change, from $x = x_1$ to $x = x_2$

is $\dfrac{\Delta C}{\Delta x} = \dfrac{C(x_2) - C(x_1)}{x_2 - x_1} = \dfrac{C_1(x_1 + \Delta x) - C_1(x_1)}{\Delta x}$, where $\Delta x \equiv x_2 - x_1$

$\xrightarrow{\Delta x \to 0}$ $C'(x_1)$ = Instantaneous Rate of change of cost with respect to the # of items produced.

Note

$$C'(x) \approx \frac{C(x+1) - C(x)}{1} = C(x+1) - C(x) \;!$$



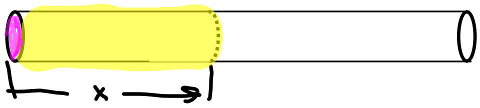
In economics, they often say the marginal cost is the cost of producing the $x+1^{th}$ **item at a given level of production.**

$C(x+1) - C(x)$

$= C(501) - C(500)$

= cost of producing the 501^{st} item.

17. The mass of the part of a metal rod that lies between its left end and a point x meters to the right is $3x^2$ kg. Find the linear density (see Example 2) when x is (a) 1 m, (b) 2 m, and (c) 3 m. Where is the density the highest? The lowest?



Mass of the shaded part is $m = m(x) = 3x^2$ Kg

Linear Density D is in units of $\frac{Kg}{m}$

$D(x) = D =$ Linear Density in $\frac{Kg}{m}$ as a function of $x =$ the # of meters from the left end of the rod, and $m = m(x) =$ mass in Kg as a function of x.

Lexicon.

$D(x) = m'(x) = 6x \ \frac{Kg}{m}$

a $D(1) = 6$

b $D(2) = 12$

c $D(3) = 18$

Density is highest @ $x=3$ and lowest @ $x=1$.

21. The force F acting on a body with mass m and velocity v is the rate of change of momentum: $F = (d/dt)(mv)$. If m is constant, this becomes $F = ma$, where $a = dv/dt$ is the acceleration. But in the theory of relativity the mass of a particle varies with v as follows: $m = m_0/\sqrt{1 - v^2/c^2}$, where m_0 is the mass of the particle at rest and c is the speed of light. Show that

$$F = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}$$

m constant $\longrightarrow$ $F = \frac{d}{dt}[mv] = m\frac{d}{dt}[v] = m\frac{dv}{dt} = ma$

where a = acceleration

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} = m_0\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Given $F = ma = m\frac{dv}{dt}$, show that

$$F = \frac{ma}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} = ma\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}$$

NOTE: Now, we're letting m vary, so $F = \frac{dm}{dt}v + m\frac{dv}{dt}$

$$\frac{d}{dt}[mv] = \frac{dm}{dt}v + m\frac{dv}{dt}$$

$$\frac{dm}{dt} = \frac{d}{dt}\left[m_0\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}\right] = m_0\left(-\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\right)\left(-\frac{2}{c^2}v\frac{dv}{dt}\right)$$

$$\frac{d}{dt}\left[1-\frac{v^2}{c^2}\right] = -\frac{2v}{c^2}\cdot\frac{dv}{dt} \qquad =$$

$$\frac{dm}{dt} = m_0\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(\frac{1}{c^2}\right)\left(v\frac{dv}{dt}\right)$$

$$= \frac{m_0 v}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}\frac{dv}{dt}$$

$$\Longrightarrow \frac{dm}{dt}v = \frac{m_0 v^2}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}\frac{dv}{dt}$$

and $m\frac{dv}{dt} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\frac{dv}{dt}$ $\longrightarrow$

$$F = \frac{dm}{dt} \cdot v + m \frac{dv}{dt}$$

$$= \frac{m_0 v^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)} \cdot \frac{dv}{dt} + \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \cdot \frac{dv}{dt}$$

$$\stackrel{?}{=} \frac{m_0 a}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

$$F = \frac{m_0 v^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \frac{dv}{dt} + \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \cdot \frac{\left(1 - \frac{v^2}{c^2}\right)^1}{\left(1 - \frac{v^2}{c^2}\right)^1} \cdot \frac{c^2}{c^2} \frac{dv}{dt}$$

$$= \frac{m_0 v^2 + m_0 \left(1 - \frac{v^2}{c^2}\right)(c^2)}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \frac{dv}{dt}$$

$$= \frac{m_0 v^2 + m_0 c^2 - m_0 \frac{v^2}{c^2} \cdot c^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \frac{dv}{dt}$$

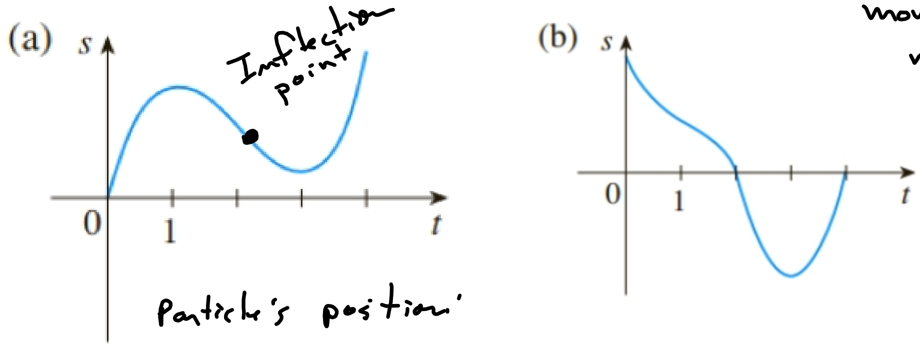
$$= \frac{m_0 c^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \cdot \frac{dv}{dt} = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \cdot \frac{dv}{dt} = \frac{m_0 a}{\left(1 - v^2/c^2\right)^{3/2}} \quad \blacksquare$$

Pretty painful.

Chain Rule stuff, algebraic simplification.

6. Graphs of the *position* functions of two particles are shown, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.

"position" in 1 dimension so particle moving left or right.



Particle's position

(a) Speeding up from $t=1$ to $t=2$ and from 3 to ∞.

When's it slowing down?

from $t=0$ to $t=1$, then from $t=2$ to $t=3$.