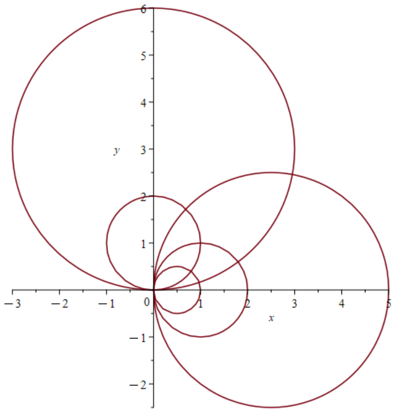
49–52 Two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are **orthogonal trajectories** of each other; that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

49. $x^2 + y^2 = r^2, \quad ax + by = 0$

50. $x^2 + y^2 = ax, \quad x^2 + y^2 = by$

51. $y = cx^2, \quad x^2 + 2y^2 = k$

52. $y = ax^3, \quad x^2 + 3y^2 = b$



$x^2+y^2=ax$

$x^2-ax+\left(\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$

$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$

$x^2+y^2=by$

$x^2+y^2-by+\left(\frac{b}{2}\right)^2=\frac{b^2}{4}$

$x^2+\left(y-\frac{b}{2}\right)^2=\left(\frac{b}{2}\right)^2$

Your job: Find y' for each & compare

$x^2+y^2=ax \longrightarrow$

$2x+2yy'=a \longrightarrow$

$2yy'=a-2x$

$y'=\frac{a-2x}{2y}=m$

$m_\perp=-\frac{1}{m}=-\frac{1}{\frac{a-2x}{2y}}=\frac{-2y}{a-2x}$

$=\frac{-2x}{2y-b}$ & $ax=by$? $x=\frac{by}{a}$?

$x^2+y^2=by$

$2x+2yy'=by'$

$2yy'-by'=-2x$

$y'(2y-b)=-2x$

$y'=\frac{-2x}{2y-b}$

when do they intersect?

$x^2+y^2 = ax$

$\Rightarrow y^2 = ax - x^2$

$y = \pm\sqrt{ax-x^2}$. Meh. This might be legit.

Hmmmm. By INSPECTION,

my thinking is

$x^2+y^2 = ax$ intersects $x^2+y^2 = by$ when

$ax = by$? Very dubious
leads nowhere

$m_\perp = \dfrac{-2y}{a-2x}$ $\qquad \dfrac{-2x}{2y-b} = \dfrac{-2x}{2}$

$x = \dfrac{by}{a}$

$m_\perp = \dfrac{-2y}{a-2\left(\frac{by}{a}\right)} = \dfrac{-2y}{\frac{a^2-2by}{a}} = -\dfrac{2ay}{a^2-2by}$

From 2^{nd} Eq'n:

$y' = \dfrac{-2x}{2y-b} = \dfrac{-2\left(\frac{by}{a}\right)}{2y-b} = \dfrac{\frac{-2by}{a}}{2y-b} = \dfrac{-2by}{2ay-ab}$

Not gettn' it.

$y = \pm\sqrt{2x - x^2}$ for 1^{st} eq'n.

Substitute $y = \sqrt{2x-x^2}$ into 2^{nd} eq'n.

$x^2 + y^2 = by$

$x^2 + \left(\sqrt{2x-x^2}\right)^2 = b\sqrt{2x-x^2}$

$x^2 + 2x - x^2 = b\sqrt{2x-x^2}$

$2x = b\sqrt{2x-x^2}$

$2^2x^2 = b^2(2x-x^2)$

$\Rightarrow 2^2x^2 = b^2 2x - b^2x^2$

$\Rightarrow (2^2+b^2)x^2 = 2b^2x$

$\Rightarrow (2^2+b^2)x^2 - 2b^2x = 0$

$\Rightarrow x((2^2+b^2)x - 2b^2) = 0$

$\Rightarrow x = 0$ or $(2^2+b^2)x - 2b^2 = 0$

$(2^2+b^2)x = 2b^2 \Rightarrow$

$x = \frac{2b^2}{2^2+b^2}$

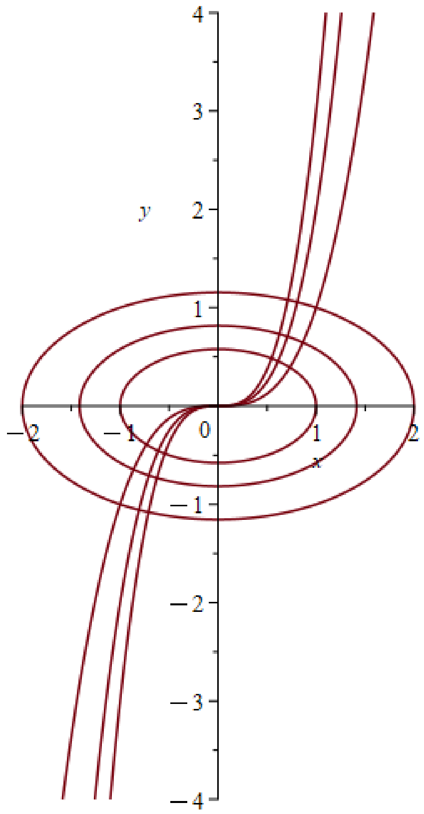
$y = \sqrt{2x-x^2} \Rightarrow$

$\boxed{y = 2\left(\frac{2b^2}{2^2+b^2}\right) - \left(\frac{2b^2}{2^2+b^2}\right)^2}$

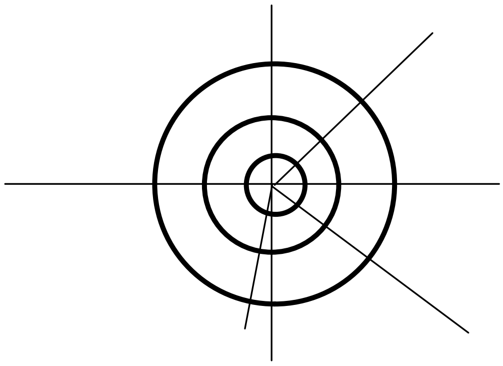
$=$

Picture for #52

Family of ellipses centered at the origin are orthogonal to cubic power functions through the origin.



$x^2+y^2=r^2$ $ax+by=0$



Circles centered @ the origin are $\perp$ to lines thru the origin.

5. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing?

6. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

7. The radius of a spherical ball is increasing at a rate of 2 cm/min. At what rate is the surface area of the ball increasing when the radius is 8 cm?

#6 Let r = radius of the sphere in mm as a function of
t = time in seconds, and
V = volume of the sphere in mm^3 as a function of r

$$V = \frac{4}{3}\pi r^3$$

We want $\left.\frac{dV}{dt}\right|_{\text{diameter} = 80\text{mm}} = \left.\frac{dV}{dt}\right|_{r=40\text{ mm}}$

$$\frac{dV}{dt} = \left(\frac{4}{3}\pi\right)\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2 \frac{dr}{dt} \Longrightarrow$$

$$\left.\frac{dV}{dt}\right|_{r=40} = 4\pi(40^2)(4) = \boxed{25600\pi \text{ mm}^3/\text{sec} = \left.\frac{dV}{dt}\right|_{r=40}}$$