Chain Rule.

$F(x) = f(g(x))$, where $g(x)$ is differentiable at a and $f(g(x))$ is differentiable at $b = g(a)$. Then

$$\frac{dF}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx} \stackrel{\text{i.e.}}{=} \frac{df}{du} \cdot \frac{du}{dx}$$

Let $u = g(x)$

$\Delta u = g(x+\Delta x) - g(x)$ is the increment of u.

$$\frac{\Delta u}{\Delta x} = \frac{g(x+\Delta x) - g(x)}{\Delta x} = g'(x) + \epsilon_1$$

This makes sense. $g(x)$ is difbl

$$\frac{\Delta u}{\Delta x} \xrightarrow{\Delta x \to 0} g'(x)$$

$\Delta u = g'(a)\Delta x + \epsilon_1 \Delta x$ NOTE $\epsilon_1 \to 0$ as $\Delta x \to 0$

$= (g'(a) + \epsilon_1)\Delta x$

Define $\epsilon_1 = 0$ at $\Delta x = 0$.

This makes ϵ_1 continuous. $y = f(u)$ is difbl @ $u = g(a)$

Now $y = f(u) \longrightarrow$

$\Delta y = f(u+\Delta u) - f(u)$

$$\frac{\Delta y}{\Delta u} = \frac{f(u+\Delta u) - f(u)}{\Delta u} = f'(g(a)) + \epsilon_2$$

$\Longrightarrow \Delta y = (f'(u) + \epsilon_2)\Delta u = (f'(u) + \epsilon_2)(g'(a) + \epsilon_2)\Delta x$

$\longrightarrow$

$$\frac{\Delta F}{\Delta x} \simeq \frac{\Delta y}{\Delta x} = (f'(g(a)) + \epsilon_2)(g'(a) + \epsilon_1) \xrightarrow{\Delta x \to 0} \frac{dy}{dx}$$

$$= F'(a) = f'(g(a))\,g'(a) = \left.\frac{df}{dg} \cdot \frac{dg}{dx}\right|_{x=a}$$

Choice of a was arbitrary, so it works $\forall\ x$.

Let $f = 5u+7 \quad = f(u)$

and $g = \sin(x) \quad = g(x)$

Then $f(g(x)) = \quad 5\sin(x) + 1$

$\frac{df}{du} = 5$

$\frac{dg}{dx} = \cos(x)$

$\longrightarrow \quad \frac{df}{du} \cdot \frac{dg}{dx} = 5\cos(x)$

59. Find all points on the graph of the function $f(x) = 2\sin x + \sin^2 x$ at which the tangent line is horizontal.

$f'(x) = 2\cos(x) + 2\sin(x)\cos(x)$

$= 2\cos(x)\left(1+\sin(x)\right) \overset{SET}{=} 0 \longrightarrow$

$F(x) = \left(\sin^2(x)\right) = \left(\sin(x)\right)^2$

$f(u) = u^2$

$u(x) = \sin(x)$

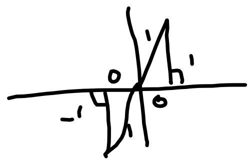
$f'(u)\,u'(x)$

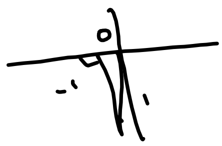
$= 2u \cdot \cos(x)$

$= 2\sin(x)\cos(x)$

$2\cos(x) = 0$ OR $\sin(x) + 1 = 0 \longrightarrow$

$\rightarrow \cos(x) = 0$ $\quad$ $\sin(x) = -1$





$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $x = \frac{3\pi}{2}$

In $[0, 2\pi)$, $x \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ = solution set

(WebAssign just wants $x = \frac{\pi}{2}, \frac{3\pi}{2}$

In general
$x \in \{ y + 2n\pi \mid y = \frac{\pi}{2}, \frac{3\pi}{2}, n \in \mathbb{Z} \}$ OR

$x \in \{ \frac{\pi}{2} + n\pi \mid n \in \mathbb{Z} \}$ as sol'ns are π apart

WebAssign wants to see

$x = \frac{\pi}{2} + n\pi$

$x = \frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi$

Both are OK in my book.

$\frac{\pi}{2} + n\pi$

$\frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi$

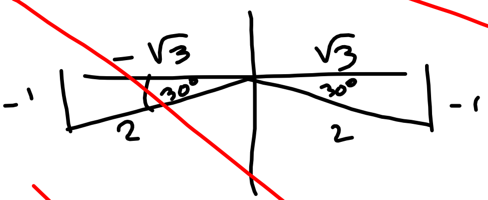
§24 #11

$f(x) = x + 2\sin(x)$ Find where $f(x)$ has a horizontal tangent.

$\Rightarrow f'(x) = 1 + 2\sin(x) \overset{\text{SET}}{=} 0$ $\to \cos(x)$

$\cancel{2\sin(x) = -1}$

$\cancel{\sin(x) = -\frac{1}{2}}$



$\cancel{180° + 30° = 210° = \frac{7\pi}{6}}$

$\cancel{360° - 30° = 330° = \frac{11\pi}{6}}$

$\frac{d}{dx}[\sin(x)] = \cos(x)$

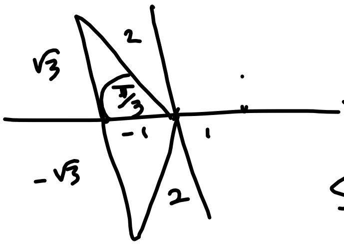
$\cancel{x = \frac{7\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi}$

$f(x) = x + 2\sin(x) \Rightarrow$

$f'(x) = 1 + 2\cos(x) \overset{\text{SET}}{=} 0 \Rightarrow$

$2\cos(x) = -1$

$\cos(x) = \frac{-1}{2}$



$\boxed{x = \frac{2\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi}$ WebAssign

Sol'n Set $= \{ y + 2n\pi \mid y = \frac{2\pi}{3}, \frac{4\pi}{3}, n \in \mathbb{Z} \}$

$= \{ y + 2n\pi \mid y \in \{\frac{2\pi}{3}, \frac{4\pi}{3}\}, n \in \mathbb{Z} \}$

60. At what point on the curve $y = \sqrt{1 + 2x}$ is the tangent line perpendicular to the line $6x + 2y = 1$?

Slope of $Ax + By = C$ is $m = -\frac{A}{B} = -\frac{6}{2} = -3$

$\longrightarrow m_{\perp} = -\frac{1}{m} = \boxed{\frac{1}{3} = m_{\perp}}$

$By = -Ax + C$

$y = -\frac{A}{B}x + \frac{C}{B}$

$y = \sqrt{2x+1} = (2x+1)^{\frac{1}{2}} \longrightarrow$

$y' = \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2) = \frac{1}{\sqrt{2x+1}} \overset{SET}{=} \frac{1}{3}$

$\frac{d}{dx}\left[\Theta^{\frac{1}{2}}\right] = \frac{1}{2}\Theta^{-\frac{1}{2}} \cdot \frac{d\Theta}{dx}$

$\longrightarrow 3 = \sqrt{2x+1} = 3$

$2x + 1 = 9$

$2x = 8$

$x = 4$ Plug in $x=4$ to find y.

$y(4) = \sqrt{2(4)+1} = \sqrt{9} = 3$

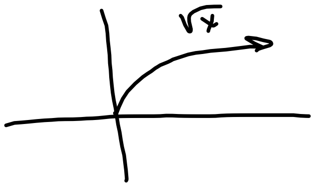
$\longrightarrow \boxed{(x, y) = (4, 3)}$

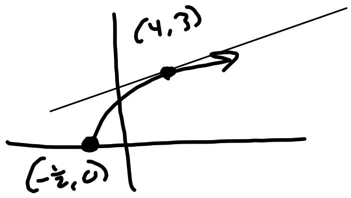
Done

Extra Enrichment.

We can draw this pretty easily

Graph: $\sqrt{2x+1} = \sqrt{2\left(x + \frac{1}{2}\right)}$





$y = \frac{1}{3}(x-4) + 3$ is the equation

61. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

$$F'(5) = \frac{d}{dx}\left[f(g(x))\right]\Big|_{x=5}$$

$$f'(g(5)) = f'(-2) = 4$$

$$g'(5) = 6$$

$$f'(g(5))\,g'(5) = 4 \cdot 6 = 24$$

$$\frac{dF}{du} \cdot \frac{du}{dx} \qquad \frac{df}{dg} \cdot \frac{dg}{dx} = f'(g(5))\,g'(5)$$

$=$

63. A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

(a) If $h(x) = f(g(x))$, find $h'(1)$.
(b) If $H(x) = g(f(x))$, find $H'(1)$.

(a) $h'(1) = f'(g(1))\, g'(1) = f'(2)(6) = 5(6) = 30$

(b) $H(x) = g(f(x)) \longrightarrow H'(1) = g'(f(1))\, f'(1) = (g'(3))(4)$

$= (9)(4) = 36$

#s 7 - 46: Find the derivative of the function.

7. $F(x) = (5x^6 + 2x^3)^4$

$\Rightarrow F'(x) = 4(5x^6+2x^3)^3(30x^5+6x^2)$

21. $g(u) = \left(\dfrac{u^3-1}{u^3+1}\right)^8 = 8\left(\dfrac{u^3-1}{u^3+1}\right)^7\left(\dfrac{3u^2(u^3+1)-(u^3-1)(3u^2)}{(u^3+1)^2}\right)$

$f = u^3-1 \Rightarrow f' = 3u^2$

$g = u^3+1 \qquad g' = 3u^2$

$\Rightarrow \left(\dfrac{f}{g}\right)' = \dfrac{f'g-fg'}{g^2} = \dfrac{3u^2(u^3+1)-(u^3-1)(3u^2)}{(u^3+1)^2}$

21. $g(u) = \left(\dfrac{u^3-1}{u^3+1}\right)^8 \Rightarrow$

$g'(u) = 8\left(\dfrac{u^3-1}{u^3+1}\right)^7\left(\dfrac{\overset{f'}{3u^2}\overset{g}{(u^3+1)} - \overset{f}{(u^3-1)}\overset{g'}{(3u^2)}}{\underset{g^2}{(u^3+1)^2}}\right)$

43. $g(x) = (2r \sin rx + n)^p$

$$g'(x) = \boxed{p\left(2r\sin(rx) + n\right)^{p-1}\left(2r\cos(rx)\cdot r\right)}$$

$$= p\left(2r\sin(rx)+n\right)^{p-1}\left(2r^2\cos(rx)\right)$$

$$= 2r^2p\left(2r\sin(rx)+n\right)^{p-1}\cos(rx)$$

44. $y = \cos^4(\sin^3 x)$

47–50 Find y' and y''.

47. $y = \cos(\sin 3\theta)$

$y' = -\sin(\sin(3\theta)) \cdot \cos(3\theta) \cdot 3 = -3\sin(\sin(3\theta))\cos(3\theta)$

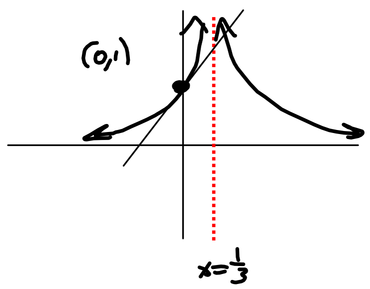
$y'' = \underbrace{-3\cos(\sin(3\theta))\, 3\cos(3\theta)}_{f'} \underbrace{\cos(3\theta)}_{g} - \underbrace{3\sin(\sin(3\theta))}_{f} \cdot \underbrace{(-3\sin(3\theta))}_{g'}$

51–54 Find an equation of the tangent line to the curve at the given point.

51. $y = (3x-1)^{-6}$, $(0, 1)$

$\frac{1}{x^2}$ shape

$\frac{1}{(3(x-\frac{1}{3}))^6}$



$y = (3x-1)^{-6}$

$y' = -6(3x-1)^{-7}(3)$

$y'(0) = -6(-1)^{-7}(3)$

$= 6(3) = 18$

$y = m(x - x_1) + y_1$

$\boxed{= 18(x-0) + 1}$

54. $y = \sin^2 x \cos x, \quad (\pi/2, 0)$

$= f g$, where

$f = \sin^2(x) \longrightarrow f'(x) = 2\sin(x)\cos(x)$

$g = \cos(x) \qquad g'(x) = -\sin(x)$

$\Rightarrow y' = f'g + fg' = \boxed{2\sin(x)\cos(x)\cos(x) - \sin^2(x)\sin(x) = y'}$

$= 2\sin(x)\cos^2(x) - \sin^3(x)$

Assume y is a function of x.

Denote $\frac{dy}{dx}$ by y'.

Differentiate xy with respect to x:

$f = x \quad f' = 1$

$g = y \quad g' = y'$ $\longrightarrow$ $(xy)' = \frac{d}{dx}[xy]$

$= f'g + fg' = 1y + xy'$

$= y + xy'$

$x^2 + y^2 = 16 \longrightarrow$

$2x + 2yy' = 0$

$f = y^2 \Longrightarrow \frac{df}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx} = 2yy'$

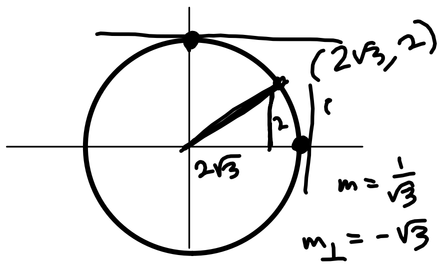
$\longrightarrow 2yy' = -2x$

$y' = \frac{-2x}{2y} = -\frac{x}{y}$ = Slope!

ⓐ $(4,0)$:

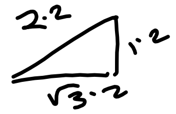
$y' = -\frac{x}{y} = -\frac{4}{0}$! $\nexists$ ✓

undefined



ⓑ $(0,4)$

$-\frac{x}{y} = -\frac{0}{4} = 0$ ✓



$y = -\frac{x}{y} = -\frac{2\sqrt{3}}{2} = -\sqrt{3}$ ✓

$x^2 + y^2 = 16$

$y^2 = 16 - x^2$

$y = \pm\sqrt{16 - x^2}$ $\nearrow$ $y = \sqrt{16 - x^2}$ top half

$= (16 - x^2)^{\frac{1}{2}}$

$y' = \frac{1}{2}(16 - x^2)^{-\frac{1}{2}}(-2x)$

$= -\frac{x}{\sqrt{16 - x^2}} = \frac{-x}{y}$

$y'(2\sqrt{3}) = \frac{-2\sqrt{3}}{\sqrt{16 - 4 \cdot 3}} =$

$= \frac{-2\sqrt{3}}{\sqrt{4}} = -\frac{2\sqrt{3}}{2} = -\sqrt{3}$ ✓