Product Rule: If $f(x)$ (f) and $g(x)$ (g) are both differentiable at x, then

$$\frac{d}{dx}\left[f(x)g(x)\right] = (fg)' = f'g + fg'$$

Proof Want to show

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \xrightarrow{h \to 0} f'g + fg'$$

$$= \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \frac{\left(f(x+h) - f(x)\right)g(x+h)}{h} + \frac{f(x)\left(g(x+h) - g(x)\right)}{h}$$

$$= \frac{f(x+h) - f(x)}{h}\, g(x+h) + f(x)\,\frac{g(x+h) - g(x)}{h}$$

$$\xrightarrow{h \to 0} f'(x)g(x) + f(x)g'(x) \quad \blacksquare$$

$(fg)' = f'g + fg'$ Learn this way.

BOOK DOES THIS

$f'g + g'f$ NOT GOOD FOR CALC III.

$$h(x) = (x^2+5x)(3x^7 - 25x^2 + x) = f(x)g(x),$$

where

$f(x) = x^2 + 5x$ $\longrightarrow$ $f' = 2x+5$

$g(x) = 3x^7 - 25x^2 + x$ $\quad$ $g' = 21x^6 - 50x + 1$

$$\longrightarrow h'(x) = f'g + fg' = (2x+5)(3x^7 - 25x^2 + x) + (x^2+5x)(21x^6 - 50x + 1)$$

DONE!

Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Proof

$$\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} \cdot \frac{g(x)}{g(x)} - \frac{f(x)}{g(x)} \cdot \frac{g(x+h)}{g(x+h)}\right]$$

$$= \frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right]$$

$$= \frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right]$$

$$= \frac{1}{h}\left[\frac{(f(x+h) - f(x))g(x)}{g(x)g(x+h)} + \frac{f(x)(g(x) - g(x+h))}{g(x)g(x+h)}\right]$$

$$= \frac{1}{g(x)g(x+h)}\left[\frac{(f(x+h) - f(x))\, g(x)}{h} + \frac{f(x)(g(x) - g(x+h))}{h}\right]$$

$$= \frac{1}{g(x)g(x+h)}\left[\frac{(f(x+h) - f(x))}{h} g(x) - f(x)\frac{(g(x+h) - g(x))}{h}\right]$$

$$\xrightarrow{h \to 0} \frac{1}{g(x)^2}\left[f'(x)g(x) - f(x)g'(x)\right]$$

$$= \frac{f'g - fg'}{g^2},$$ as desired.

$$h(x) = \frac{(x^7 - 5x^4 + 1)}{3x^5 + 5x^3 - 1} = \frac{f}{g}$$

$$f = x^7 - 5x^4 + 1 \qquad f' = 7x^6 - 20x^3$$

$$g = 3x^5 + 5x^3 + 1 \longrightarrow g' = 15x^4 + 15x^2$$

$$\longrightarrow h' = \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$= \frac{(7x^6 - 20x^3)(3x^5 + 5x^3 + 1) - (x^7 - 5x^4 + 1)(15x^4 + 15x^2)}{(35x^5 + 5x^3 + 1)^2}$$

Just stop!

$$h(x) = \frac{(x^7 - 5x^4 + 1)}{3x^5 + 5x^3 - 1} = \frac{f}{g}$$

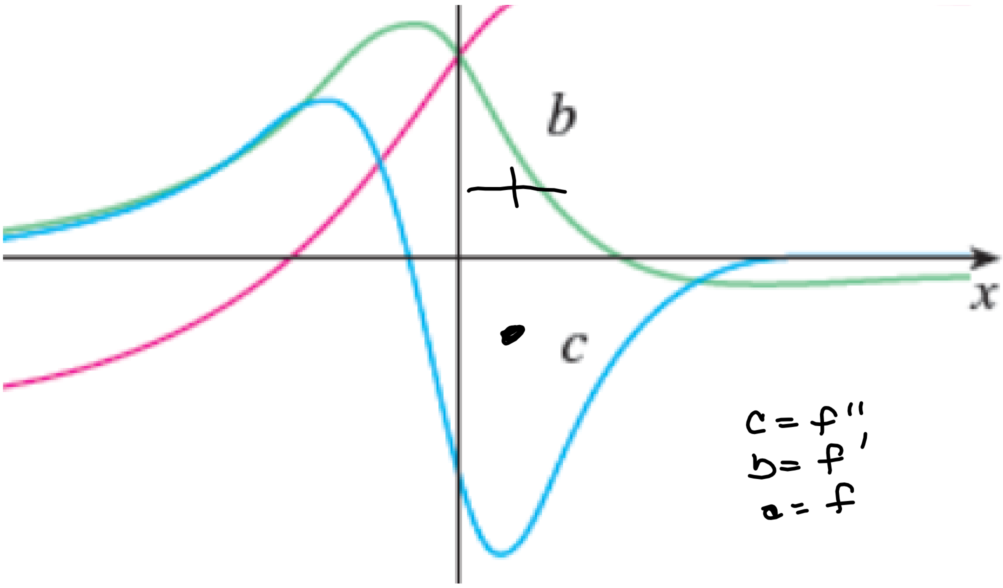
$f = x^7 - 5x^4 + 1$ $\qquad f' = 7x^6 - 20x^3$

$g = 3x^5 + 5x^3 + 1 \longrightarrow g' = 15x^4 + 15x^2$

$$\longrightarrow h' = \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$= \frac{(7x^6 - 20x^3)(3x^5 + 5x^3 + 1) - (x^7 - 5x^4 + 1)(15x^4 + 15x^2)}{(35x^5 + 5x^3 + 1)^2}$$

Just stop!



c = f''
b = f'
a = f

Homework

State givens

state question

Define variables needed/used for word problems
(Lexicon)

systematically work out the solution.

state symbolic answer.
Circle it $\boxed{x=5}$

Proof

State result

Work to the result.

§ 2.3 Trig Derivatives

$\frac{d}{dx}[\sin(x)] = \cos(x)$

or

$f(x) = \sin(x) \Longrightarrow f'(x) = \cos(x)$

A little sketchy about saying

$(\sin(x))' = \cos(x)$, but it's legit.

Proof write the difference quotient

$$\frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

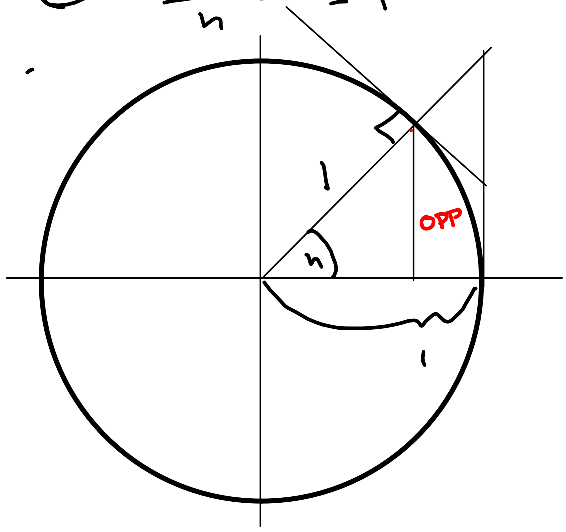
$$= \frac{\sin(x)(\cos(h) - 1)}{h} + \frac{\sin(h)\cos(x)}{h}$$

Where we stalled

want to get $\cos(x)$ out of this.

so, we want ② $\frac{\cos(h) - 1}{h} \xrightarrow{h \to 0} 0$

① $\frac{\sin(h)}{h} = 1$

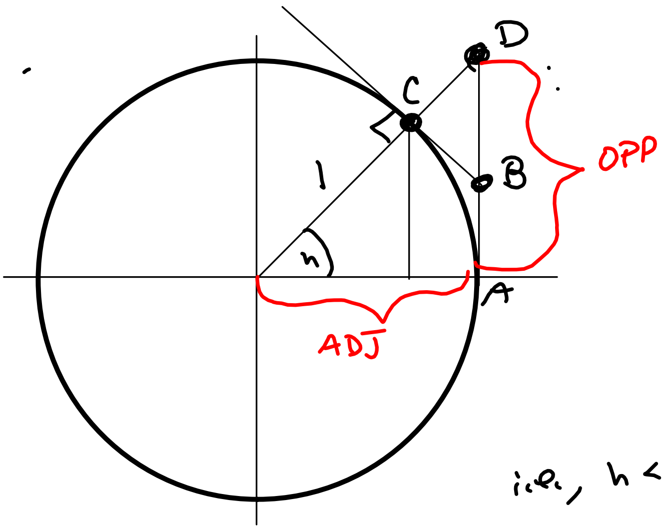


Goal: Squeeze $\frac{\sin(h)}{h}$ between $\cos(h)$ & 1

$\sin(h) = \frac{\text{OPP}}{1} <$ arc length $= h$

i.e. $\sin(h) < h$, i.e.,

$$\boxed{\frac{\sin(h)}{h} < 1}$$



arc length $= h$

$< |\overline{AB}| + |\overline{BC}|$

$< |AB| + |\overline{BD}|$

$= \frac{|AB| + |BD|}{1}$

$= \frac{\text{OPP}}{\text{ADJ}} = \tan(h)$

i.e., $h < \tan(h) = \frac{\sin(h)}{\cos(h)}$

$$\longrightarrow \boxed{\cos(h) < \frac{\sin(h)}{h}}$$

$$\therefore \quad \cos(h) < \frac{\sin(h)}{h} < 1$$

$$1 \le \lim_{h\to 0} \frac{\sin(h)}{h} \le 1 \qquad (\cos(h) \xrightarrow{h\to 0} 1, \quad 1 \xrightarrow{h\to 0} 1)$$

$$\therefore \quad \boxed{\lim_{h\to 0} \frac{\sin(h)}{h} = 1}$$

(2) Need $\dfrac{\cos(h)-1}{h} \xrightarrow{h\to 0} 0$

$$\left(\frac{\cos(h)-1}{h}\right)\left(\frac{\cos(h)+1}{\cos(h)+1}\right) = \frac{\cos^2(h)-1}{h(\cos(h)+1)} = \frac{-\sin^2(h)}{h(\cos(h)+1)}$$

$$= -\frac{\sin(h)}{h}\left(\frac{\sin(h)}{\cos(h)+1}\right) \xrightarrow{h\to 0} -1 \cdot \frac{0}{1} = 0$$

Where we stalled in main proof for $\frac{d}{dx}[\sin(x)] = \cos(x)$:

$$= \frac{\sin(x)(\cos(h)-1)}{h} + \frac{\sin(h)\cos(x)}{h} \xrightarrow{h\to 0}$$

$$\sin(x)(0) + 1 \cdot \cos(x) = \cos(x) \quad \blacksquare$$

$$\frac{\cos(x+h)-\cos(x)}{h}$$

$$= \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$

$$= \frac{\cos(x)}{1}\cdot\left(\frac{\cos(h)-1}{h}\right)-\sin(x)\cdot\frac{\sin(h)}{h}) \xrightarrow{h\to 0} -\sin(x)$$

$$\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$$

$$\frac{d}{dx}\left[\tan(x)\right] = \sec^2(x)$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{f}{g}$$

$$f' = \cos(x)$$
$$g' = -\sin(x)$$

$$\Longrightarrow \quad \frac{f'g - fg'}{g^2} = \frac{\cos(x)\cos(x)-\sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

$$\boxed{\frac{d}{dx}\left[\tan(x)\right] = \sec^2(x)}$$