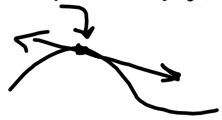
1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

$$(a, f(a)) = (x_1, y_1)$$

provided that this limit exists.



The tangent line has equation

$$y = m(x - x_1) + y_1$$

Book: $y - y_1 = m(x - x)$

$$L(x) = f'(x_1)(x - x_1) + f(x_1)$$

or $L_a(x)$ to mean the linearization of $f(x)$ @ $x = a$. $L_a(x) = f'(a)(x-a) + f(a)$

$\boxed{E}$ $f(x) = x^2$, $a = 1$. $f(a) = f(1) = 1^2 = 1$

$$\frac{f(x) - f(a)}{x - a} = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1} = x + 1 \xrightarrow{x \to 1} 1 + 1 = \boxed{2 = m}$$

$(x \neq 1)$

$$y = m(x - x_1) + y_1$$

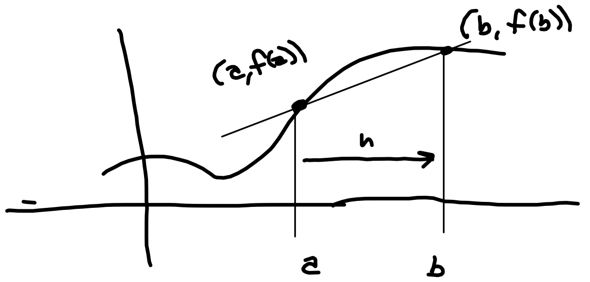
$$L(x) = f'(1)(x - 1) + f(1)$$

$$\boxed{L(x) = 2(x-1) + 1}$$ STOP!

Much better than finding b & being trapped in $y = mx + b$ mode.

$$\text{Average Velocity} = \frac{\text{Displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{h} = \frac{f(b)-f(a)}{b-a}$$

$$\text{Instantaneous Velocity} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$



Avg velocity on $[a,b] = [a, a+h]$, where $h = b-a$

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

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$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$x = a+h$

$$f'(x) = \lim_{c \to x} \frac{f(c) - f(x)}{c - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$x+h - x = h$

$s(t) = \frac{1}{2}gx^2 + v_0x + s_0$ From Week 2's.

$f(x) = -16x^2 + 50x + 10$

I said $\frac{50\,mi}{hr} = v_0$ but this is in the wrong units. SHOULD be $50\,\frac{ft}{sec}$ for this to be the model.

$a = 1,\ h = .5, .1, .001$

$$\frac{f(1+h) - f(1)}{h} = \frac{-16(1+h)^2 + 50(1+h) + 10 - (-16(1)^2 + 50(1) + 10)}{h}$$

$$= \frac{-16(1+2h+h^2) + 50 + 50h + 10 + 16 - 50 - 10}{h} \quad = \cancel{\frac{-16 - 32h - 16h^2 + 16 - 50 - 10}{h}} \text{ New P}$$

$$= \frac{-16 - 32h - 16h^2 + 50 + 50h + 10 + 16 - 50 - 10}{h}$$

$$= \frac{-32h - 16h^2 + 50h}{h} = \frac{h(-16h + 18)}{h} = -16h + 18 \quad (h \neq 0)$$

(a) Slope $= -8 + 18 = \boxed{10\ \frac{ft}{s}}$

(b) Slope $= -16(.1) + 18 = -1.6 + 18 = \boxed{16.4\ \frac{ft}{s}}$

(c) Slope $= -16(.001) + 18 = -.0016 + 18 = \boxed{17.9984\ \frac{ft}{s}}$

Example 4 Find $f'(a)$ for $f(x) = x^2 - 7x + 2$

$$\frac{f(a+h) - f(a)}{h} = \frac{(a+h)^2 - 7(a+h) + 2 - (a^2 - 7a + 2)}{h}$$

$$= \frac{a^2 + 2ah + h^2 - 7a - 7h + 2 - a^2 + 7a - 2}{h}$$

$$= \frac{2ah + h^2 - 7h}{h} = \frac{h(2a + h - 7)}{h} = \underset{h \neq 0}{2a + h - 7} \xrightarrow{h \to 0} 2a - 7$$

$$\frac{f(x) - f(a)}{x - a} = \frac{x^2 - 7x + 2 - (a^2 - 7a + 2)}{x - a}$$

$$= \frac{x^2 - 7x + 2 - a^2 + 7a - 2}{x - a} = \frac{x^2 - a^2 - 7x + 7a}{x - a}$$

$$= \frac{(x-a)(x+a) - 7(x-a)}{x-a} = \frac{(x-a)(x + a - 7)}{x - a} = \underset{x \neq a}{x + a - 7} \xrightarrow{x \to a} \boxed{2a - 7 = f'(a)}$$

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

Book:

$y - f(a) = f'(a)(x-a)$

BETTER WAY

$y = f'(a)(x-a) + f(a) = L(x)$

Find eq'n of tangent line to $f(x) = x^2 - 7x + 2$

(a) $a = 4$ $\qquad f(x) = x^2 - 7x + 2$

Recall $f'(a) = 2a - 7 \longrightarrow f'(4) = 2(4) - 7 = 1$

$f(a) = 4^2 - 7(4) + 2 = 16 - 28 + 2 = -10$

$y = f'(4)(x-4) + f(4)$

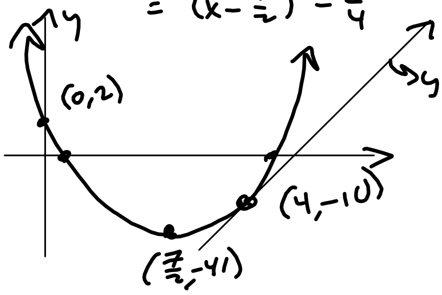
$= \boxed{1(x-4) - 10 = y}$ Perfect.

Draw the picture:

We draw the picture for this situation.

$x^2 - 7x + 2 = x^2 - 7x + \left(\frac{7}{2}\right)^2 - \frac{49}{4} + \frac{8}{4}$

$= \left(x - \frac{7}{2}\right)^2 - \frac{41}{4}$



$y = (x-4) - 10 = x - 14$, but why? (Why bother?)

Another way to find $x = \frac{7}{2}$ vertex:

$f'(x) = 2x - 7 \overset{\text{SET}}{=} 0 \longrightarrow$

$2x = 7$

$\longrightarrow x = \frac{7}{2} = x$-coord of vertex

Now find $f\left(\frac{7}{2}\right) = \left(\frac{7}{2}\right)^2 - 7\left(\frac{7}{2}\right) + 2$

$= \frac{49}{4} - \frac{49}{2} + \frac{8}{4}$

$= \frac{57}{4} - \frac{98}{4} = -\frac{41}{4}$

$(h,k) = $ vertex

$= \left(\frac{7}{2}, -\frac{41}{4}\right)$

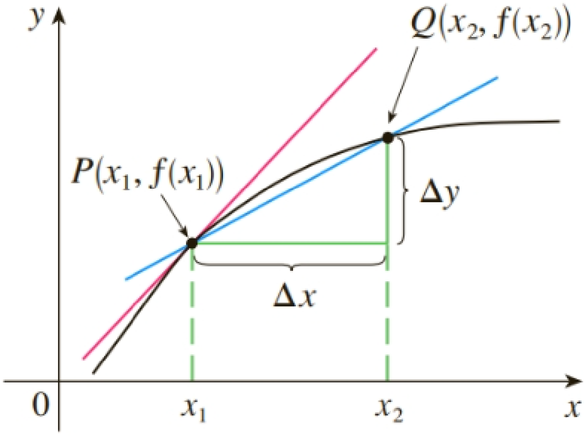
"Rate of change" is the more general term for "speed" and "velocity."

Δy = change in y

Δx = " " x

$\Delta x = x_2 - x_1$

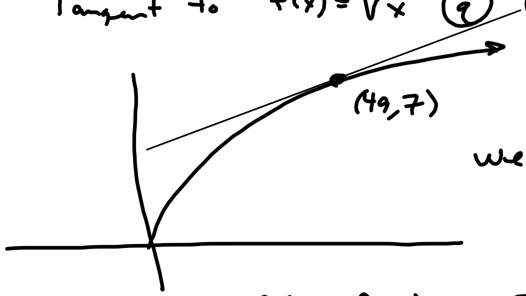
$\Delta y = f(x_2) - f(x_1)$



Average rate of change $= \dfrac{\Delta y}{\Delta x} \xrightarrow{\Delta x \to 0}$ Instantaneous rate of change.

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

Tangent to $f(x)=\sqrt{x}$ @ $(49,7)$



we need $f'(49)$

oops!

$$\frac{f(x)-f(49)}{x-49} = \frac{\sqrt{x}-\sqrt{49}}{x-49} = \left(\frac{\sqrt{x}-7}{x-49}\right)\left(\frac{\sqrt{x}+7}{\sqrt{x}+7}\right)$$

$$= \frac{x-49}{(x-49)(\sqrt{x}+7)} \underset{x\neq 49}{=} \frac{1}{\sqrt{x}+7} \xrightarrow{x\to 49} \frac{1}{\sqrt{49}+7} = \frac{1}{7+7} = \frac{1}{14} = f'(49)$$

$$\boxed{L(x) = y = \frac{1}{14}(x-49)+7}$$

STOP!

$$= f'(49)(x-49)+f(49)$$

Written Assignment #4 (Week 4)

$$f(x)=\begin{cases} x\sin\left(\frac{\pi}{x}\right) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$ is cnt^s but NOT differentiable @ $x=0$

We showed it was cnt^s in the notes.

How do we show/test its differentiability @ $x=0$

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-0}{h} = \lim_{h\to 0}\frac{h\sin\left(\frac{\pi}{h}\right)}{h}$$

$= \lim_{h\to 0}\left(\sin\left(\frac{\pi}{h}\right)\right)$ which we know doesn't exist.

What about $x^2\sin\left(\frac{\pi}{x}\right)=g(x)$?

$$g(x)=\begin{cases} x^2\sin\left(\frac{\pi}{x}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$$ is cnt^s @ $x=0$ by the

same squeeze theorem argument we used for $f(x)$

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

$$-x^2 \leq x^2\sin\left(\frac{\pi}{x}\right) \leq x^2$$

$$\underset{x\to 0}{\downarrow}\quad\quad \underset{x\to 0}{\downarrow}\quad\quad \underset{x\to 0}{\downarrow}$$

$$0 \quad\quad 0 \quad\quad 0$$

$\Rightarrow \lim_{x\to 0} g(x) = g(0) = 0$! ✓

Differentiability of $g(x)$ @ $x=0$
(It's smooth everywhere else.)

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \frac{h^2 \sin\left(\frac{\pi}{h}\right) - 0}{h}$$

$$= \frac{h^2 \sin\left(\frac{\pi}{h}\right)}{h} = h\sin\left(\frac{\pi}{h}\right) \xrightarrow{h\to 0} 0$$ (From $x\sin\left(\frac{\pi}{x}\right)$ is cnt's talk)

It exists! $f'(0)$ exists!
$f'(0)=0$.