§1.7 Wrap-Up

Linear functions.

Claim:

$\lim_{x \to 2} (5x-3) = 7$ Required.

Proof:

Let $\varepsilon > 0$. Define $\delta = \frac{\varepsilon}{5}$. Then $0 < |x-2| < \delta \implies$

$|f(x) - L| = |5x-3-7| = |5x-10| = 5|x-2| < 5\delta$

$= 5 \cdot \frac{\varepsilon}{5} = \varepsilon$ ■

Bonus: Polynomials of higher degree.

Big trick is assuming $\delta \le 1$

<u>Claim:</u>

$$\lim_{x \to 3} (x^2 - 2x + 1) = 4$$

<u>Scratch</u> Assume $\delta \le 1$

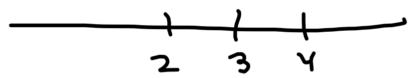
Want $|x^2 - 2x + 1 - 4| = |x^2 - 2x - 3| = |x+1||x-3|$

$< \underbrace{|x+1|}\,\delta$

Need an upper bound on $|x+1|$

$\delta \le 1$ & $\lim_{x \to 3}$ $\longrightarrow$

$\delta = 1$
$x \in (2, 4)$



$2 < x < 4$

$2+1 = 3 < x+1 < 4+1 = 5$

$\Longrightarrow |x+1| < 5$!

<u>Write Proof</u>

Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{5}\}$.

Then $0 < |x-3| < \delta \Longrightarrow$

$|x^2 - 2x + 1 - 4| = |x^2 - 2x - 3| = |x+1||x-3| < 5\delta$

$\le 5 \cdot \frac{\varepsilon}{5} = \varepsilon$ ■

$\lim_{x \to 5} (x^2 - 4x + 3) = 8$

Scratch:

want $|f(x) - L| = |x^2 - 4x + 3 - 8| = |x^2 - 4x - 5|$

$= \underbrace{|x-5|}_{< \delta} \underbrace{|x+1|} < \varepsilon$

Need an upper bound on $|x+1|$

Assume $\delta \leq 1$. then

$4 < x < 6 \rightarrow$

$5 < x+1 < 6 \rightarrow$

$|x+1| < 6$

Write the proof.

Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{6}\}$

Then $0 < |x-5| < \delta \rightarrow$

$|f(x) - L| = |x^2 - 4x + 3 - 8| = |x^2 - 4x - 5| = |x+1||x-5|$

$< 6\delta \leq 6 \cdot \frac{\varepsilon}{6} = \varepsilon$ ∎

$\lim_{x\to a} \sqrt{x} = \sqrt{a} \quad (a>0)$

Scratch: Assume $\frac{1}{2}a < x < \frac{3}{2}a$

$$\sqrt{x}-\sqrt{a} = \frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{\sqrt{x}+\sqrt{a}} = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{|x-a|}{\sqrt{\frac{1}{2}a}+\sqrt{a}} < \varepsilon$$

iff $|x-a| < \left(\sqrt{\tfrac{1}{2}a}+\sqrt{a}\right)\varepsilon$

<u>Write proof</u>

Let $\varepsilon>0$ be given. Define $\delta = \min\left\{\frac{a}{2}, \left(\sqrt{\tfrac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$.

Then $0<|x-a|<\delta \longrightarrow$

$$|\sqrt{x}-\sqrt{a}| < \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{|x-a|}{\sqrt{\frac{1}{2}a}+\sqrt{a}} < \frac{\delta}{\sqrt{\frac{1}{2}a}+\sqrt{a}} \le \frac{\left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon}{\sqrt{\frac{1}{2}a}+\sqrt{a}}$$

$$= \varepsilon \blacksquare$$

How I would handle proof of #19:

$$\lim_{x \to 2} (x^3 - 2x^2 + 5x - 9) = 1$$

Scratch

want $|x^3 - 2x^2 + 5x - 9 - 1| = |x^3 - 2x^2 + 5x - 10|$

$= |x^2+5||x-2|$

$x-2 \overline{) \, x^3 - 2x^2 + 5x - 10}$ ugh.

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 5 & -10 \\ & & 2 & 0 & 10 \\ \hline & 1 & 0 & 5 & r0 \end{array}$$

Need a bound on x^2+5

$\delta \leq 1$ & $x \to 2$

$\Rightarrow 1 < x < 3$

$\Rightarrow 1^2 < x^2 < 3^2$

$1 < x^2 < 9$

$6 < x^2+5 < 14$

$\Rightarrow x^2+5 = |x^2+5| < 14$

Write Proof

Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{14}\}$.

Then $0 < |x-2| < \delta \Rightarrow |x^3 - 2x^2 + 5x - 9 - 1|$

$= |x^3 - 2x^2 + 5x - 10| = |x^2+5||x-2| < 14\delta \leq 14 \cdot \frac{\varepsilon}{14} = \varepsilon$ ∎

$(x-1)(x^2-4x-5)$

$= x^3-4x^2-5x$

$\quad -x^2+4x+5$

$= x^3-5x^2-x+5$

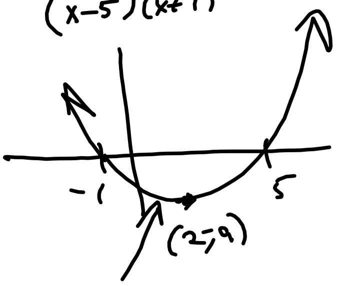
Claim $\lim_{x\to 1}(x^3-5x^2-x) = 5$

Scratch

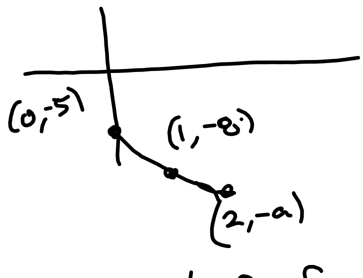
$$|x^3-5x^2-x-5| = \underbrace{|x^2-4x-5|}_{x^2-4x-5}\,\underbrace{|x-1|}_{<\delta} < \varepsilon \quad \text{want}$$

Need a bound on this.

$x^2-4x-5 = (x-5)(x+1)$



We're in here.



x^2-4x-5

$= x^2-4x+2^2-4-5$

$= (x-2)^2-9$

$\delta \le 1 \Longrightarrow 0 < x < 2$

$\Longrightarrow |x^2-4x-5| \le 9$

Write Proof

Let $\varepsilon > 0$. Define $\delta = \min\{1, \frac{\varepsilon}{9}\}$. Then $0<|x-1|<\delta$

$\Longrightarrow |f(x)-L| = |x^3-5x^2-x-5| = |x^2-4x-5||x-1|$

$< 9\delta \le 9\cdot\frac{\varepsilon}{9} = \varepsilon$ ■

$$\lim_{x \to 4} \left(\tfrac{1}{4}x + 4\right) = 5$$

Let $\varepsilon > 0$. We need $\delta > 0$ such that

$0 < |x-4| < \delta \implies |\tfrac{1}{4}x + 4 - 5| < \varepsilon$

iff $|\tfrac{1}{4}x - 1| < \varepsilon$

iff $\tfrac{1}{4}|x-4| < \varepsilon$

iff $|x-4| < 4\varepsilon$

Choose $\delta = 4\varepsilon$, etc.

$$\lim_{x \to -3} (-2x+1) = 7$$

Pf

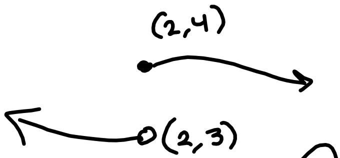
Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{2}$. Then $0 < |x-(-3)| < \delta$ implies $|-2x+1-7| = |-2x-6| = |-2||x+3|$

$= 2|x+3| < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon$ ▨

Section 1.8 - Continuity

Intermediate Value Theorem.

f is cnt^s @ $x = c$ iff $\lim\limits_{x \to c} f(x) = f(c)$

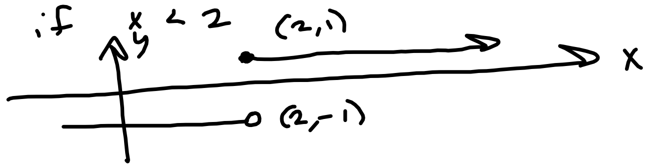


Jump discontinuity @ $x = 2$

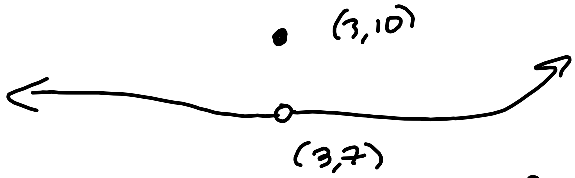
$$\lim_{x \to 2^-} f(x) = 3 \neq 4 = \lim_{x \to 2^+} f(x)$$

$$f(x) = \frac{x-2}{|x-2|} = \begin{cases} \frac{x-2}{x-2} & \text{if } x-2 \geq 0 \\ \frac{x-2}{-(x-2)} & \text{if } x-2 < 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \geq 2 \\ -1 & \text{if } x < 2 \end{cases}$$



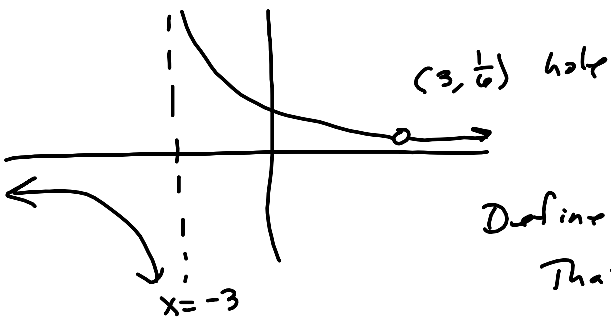
Removable Discontinuity



Removable: Define $f(3) = 7$.

$$f(x) = \frac{x-3}{x^2-9} = \frac{x-3}{(x-3)(x+3)} = \frac{1}{x+3}$$

$(x \neq 3)$

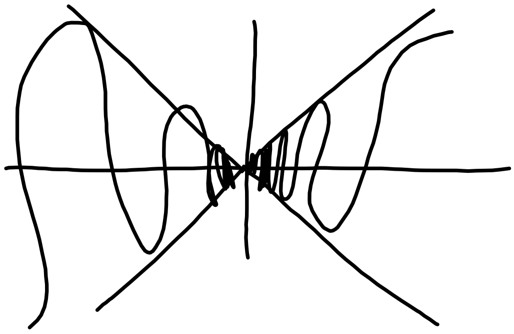


$$\text{Define } f(x) = \begin{cases} \frac{x-3}{x^2-9} & \text{if } x \neq 3 \\ \frac{1}{6} & \text{if } x = 3 \end{cases}$$

That fills in the hole.

$f(x) = x\sin(\frac{1}{x})$ has an issue (undefined) @ $x=0$

Define $g(x) \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$!



$f(x)$ is basically continuous on its domain.

Rational, Radical, polynomial, trigs,

Basically any function you can write (or think of) is cnts on its domain.

Special: Step function (Greatest Integer Function)

$f(x) =$ greatest integer less than or equal to x.

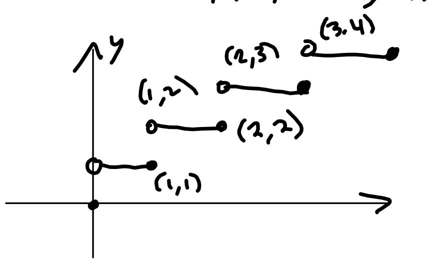
$f(1.7) = 1$,

$f(1.999) = 1$

$f(1) = 1$

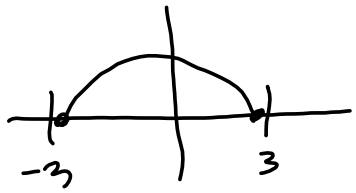
$f(2) = 2, \ldots$

$f(x) = [\![x]\!]$



A bunch of jump discontinuities.

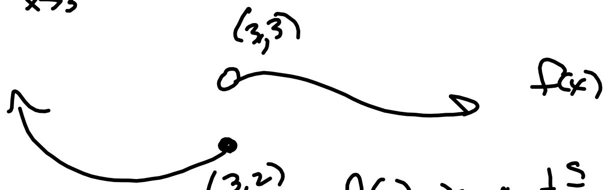
$$f(x) = \sqrt{9-x^2} = y$$
$$9 - x^2 = y^2$$
$$9 = x^2 + y^2$$



we say that $f(x)$ is cnt^s on $[-3,3]$, where it's understood that the limits @ the boundaries are one-sided.

$$\lim_{x \to -3^+} f(x) = 0 = f(-3)$$

$$\lim_{x \to 3^-} f(x) = 0 = f(3)$$



$f(x)$ is cnt^s on $(-\infty, 3]$

Also cnt^s on $(3, \infty)$

But wait! this says

f cnt^s on $(-\infty, 3] \cup (3, \infty)$

$= (-\infty, \infty)$?!

I'm not very happy with that.

Usually, with this, I end up using open intervals. This is especially true when we graduate to differentiability (existence of the derivative at *x*.)

If $f(x)$ & $g(x)$ are cnt^{s} @ $x=c$, then
so is $\sqrt[n]{f}$, $f \pm g$, $\frac{f}{g}$ (provided $g(c) \neq 0$)

<u>Composition of functions:</u>

If $g(x)$ is cnt^{s} @ $x=c$ and
$f(x)$ is cnt^{s} @ $g(c)$, then
$(f \circ g)(x)$ is cnt^{s} @ $x=c$.

$\sin\left(\frac{1}{x}\right)$ is cnt^{s} $\forall\, x \neq 0$

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$x\sin\left(\frac{1}{x}\right)$ is cnt^{s} $\forall\, x \neq 0$

What's $\lim\limits_{x \to 0} x\sin\left(\frac{1}{x}\right)$?

$$-1 \leq \sin\left(\tfrac{1}{x}\right) \leq 1$$

$x \to 0^{+} \Longrightarrow \quad -x \leq x\sin\left(\frac{1}{x}\right) \leq x \quad$ (if $x > 0$)

$x \to 0^{-} \quad -x \geq x\sin\left(\frac{1}{x}\right) \geq x \quad$ (if $x < 0$)

$$\underset{\substack{x \to 0}}{\downarrow}\, 0 \qquad \underset{\substack{x \to 0}}{\downarrow}\, 0 \qquad \underset{\substack{x \to 0}}{\downarrow}\, 0$$

Squeeze Theorem!

Intermediate Value Theorem

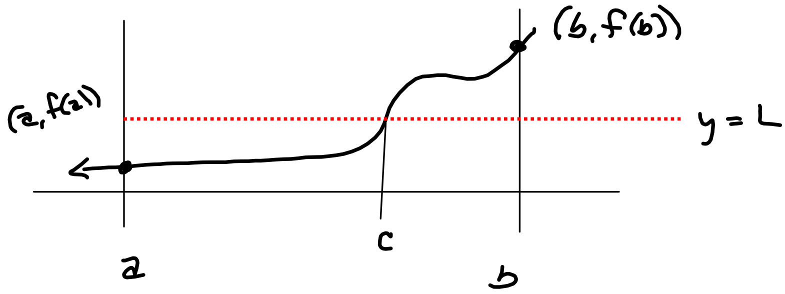
~~Sf~~

If $f(x)$ is cnts on $[a,b]$, $f(a) < f(b)$

and L is any number such that $f(a) < L < f(b)$,

$\ni$

then $\exists\ c \in (a,b) \ni f(c) = L$.



PROVE THAT $f(x) = x^4 - 7x^2 + 3x - 1$ has a zero in $[-1, 4]$.

f is cnts b/c it's a polynomial.

$f(-1) = (-1)^4 - 7(-1)^2 + 3(-1) - 1$

$= 1 - 7 - 3 - 1 = -10 < 0$

~~f(2)~~ = $f(4)$:

$$\begin{array}{r|rrrrr} 4 & 1 & 0 & -7 & 3 & -1 \\ & & 4 & 16 & 36 & 156 \\ \hline & 1 & 4 & 9 & 39 & 155 = f(4) \end{array}$$

$f(4) > 0$
$f(-1) < 0$ $\Rightarrow \exists\ c \in (-1, 4) \ni f(c) = 0$!