§1.7 Wrap-Up

Linear functions.

Claim:

$\lim_{x \to 2} (5x-3) = 7$ Required.

Proof:

Let $\varepsilon > 0$. Define $\delta = \frac{\varepsilon}{5}$. Then $0 < |x-2| < \delta \Rightarrow$

$|f(x) - L| = |5x-3-7| = |5x-10| = 5|x-2| < 5\delta$

$= 5 \cdot \frac{\varepsilon}{5} = \varepsilon$ ∎

Bonus: Polynomials of higher degree.

Big trick is assuming $\delta \le 1$

Claim:

$$\lim_{x \to 3} (x^2 - 2x + 1) = 4$$

Scratch Assume $\delta \le 1$

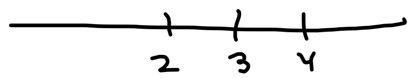
Want $|x^2 - 2x + 1 - 4| = |x^2 - 2x - 3| = |x+1||x-3|$

$< \underbrace{|x+1|}\,\delta$

Need an upper bound on $|x+1|$

$\delta \le 1$ & $\lim_{x \to 3}$ $\longrightarrow$

$\delta = 1$
$x \in (2, 4)$



$2 < x < 4$

$2+1 = 3 < x+1 < 4+1 = 5$

$\Longrightarrow |x+1| < 5$!

Write Proof

Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{5}\}$.

Then $0 < |x-3| < \delta \Longrightarrow$

$|x^2 - 2x + 1 - 4| = |x^2 - 2x - 3| = |x+1||x-3| < 5\delta$

$\le 5 \cdot \frac{\varepsilon}{5} = \varepsilon$ $\blacksquare$

$\lim_{x \to 5} (x^2 - 4x + 3) = 8$

Scratch:

want $|f(x) - L| = |x^2 - 4x + 3 - 8| = |x^2 - 4x - 5|$

$= \underbrace{|x-5|}_{< \delta} \underbrace{|x+1|} < \varepsilon$

Need an upper bound on $|x+1|$

Assume $\delta \le 1$. then

$4 < x < 6 \longrightarrow$

$5 < x+1 < 6 \longrightarrow$

$|x+1| < 6$

Write the proof.

Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{6}\}$

Then $0 < |x-5| < \delta \longrightarrow$

$|f(x) - L| = |x^2 - 4x + 3 - 8| = |x^2 - 4x - 5| = |x+1||x-5|$

$< 6\delta \le 6 \cdot \frac{\varepsilon}{6} = \varepsilon$ ∎

$$\lim_{x\to a} \sqrt{x} = \sqrt{a} \quad (a>0)$$

Scratch: Assume $\frac{1}{2}a < x < \frac{3}{2}a$

$$\sqrt{x}-\sqrt{a} = \frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{\sqrt{x}+\sqrt{a}} = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{|x-a|}{\sqrt{\frac{1}{2}a}+\sqrt{a}} < \varepsilon$$

iff $|x-a| < \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$

<u>Write proof</u>

Let $\varepsilon > 0$ be given. Define $\delta = \min\left\{\frac{a}{2}, \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$.

Then $0 < |x-a| < \delta \longrightarrow$

$$|\sqrt{x}-\sqrt{a}| < \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{|x-a|}{\sqrt{\frac{1}{2}a}+\sqrt{a}} < \frac{\delta}{\sqrt{\frac{1}{2}a}+\sqrt{a}} \le \frac{\left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon}{\sqrt{\frac{1}{2}a}+\sqrt{a}}$$

$$= \varepsilon \quad \blacksquare$$

How I would handle proof of #19:

$$\lim_{x\to 2}(x^3-2x^2+5x-9)=1$$

Scratch

want $|x^3-2x^2+5x-9-1| = |x^3-2x^2+5x-10|$

$= |x^2+5||x-2|$

$x-2 \,\overline{)\, x^3-2x^2+5x-10}$ ugh.

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 5 & -10 \\ & & 2 & 0 & 10 \\ \hline & 1 & 0 & 5 & r\,0 \end{array}$$

Need a bound on x^2+5

$\delta \le 1$ & $x \to 2$

$\Rightarrow 1 < x < 3$

$\Rightarrow 1^2 < x^2 < 3^2$

$1 < x^2 < 9$

$6 < x^2+5 < 14$

$\Rightarrow x^2+5 = |x^2+5| < 14$

Write Proof

Let $\varepsilon>0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{14}\}$.

Then $0<|x-2|<\delta \Rightarrow |x^3-2x^2+5x-9-1|$

$= |x^3-2x^2+5x-10| = |x^2+5||x-2| < 14\delta \le 14\cdot\frac{\varepsilon}{14} = \varepsilon$ ∎

$(x-1)(x^2-4x-5)$

$= x^3-4x^2-5x$
$\quad -x^2+4x+5$

$= x^3-5x^2-x+5$

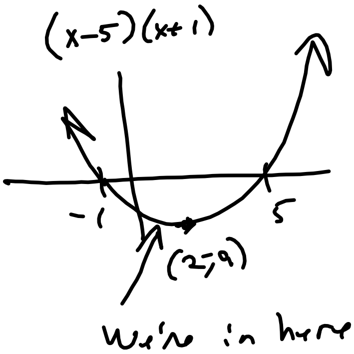
<u>Claim</u> $\lim_{x\to 1}(x^3-5x^2-x) = 5$

<u>Scratch</u>

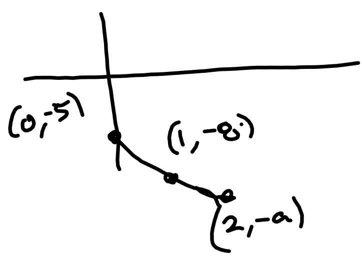
$|x^3-5x^2-x-5| = \underbrace{|x^2-4x-5|}_{x^2-4x-5}\ \underbrace{|x-1|}_{<\delta}$ Want $<\varepsilon$

x^2-4x-5 Need a bound on this.

$x^2-4x-5 = (x-5)(x+1)$



x^2-4x-5
$= x^2-4x+2^2-4-5$
$= (x-2)^2-9$



$\delta \le 1 \Rightarrow 0 < x < 2$
$\Rightarrow |x^2-4x-5| \le 9$

<u>Write Proof</u>

Let $\varepsilon > 0$. Define $\delta = \min\{1, \frac{\varepsilon}{9}\}$. Then $0 < |x-1| < \delta$

$\Rightarrow |f(x)-L| = |x^3-5x^2-x-5| = |x^2-4x-5||x-1|$

$< 9\delta \le 9\cdot\frac{\varepsilon}{9} = \varepsilon$ ■