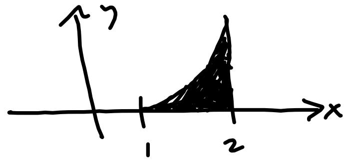
2410 **Week 14** **Mills**

1 2 we find the area bounded by:

$f(x) = 2x^3 - 7x^2 + 5x$ & $g(x) = x^3 - x^2 - 3x$, $x = 0$, $x = 5$

Sols Sketch & find area. Sketch:



$$|f(x) - g(x)| = |2x^3 - 7x^2 + 5x - x^3 + x^2 + 3x|$$

$$= |x^3 - 6x^2 + 8x|$$

$$= |x(x^2 - 6x + 8)| = |x(x-4)(x-2)|$$

$-$ $+$ $-$ $+$

0 2 4

$$\Rightarrow \int_0^5 |f(x) - g(x)|\,dx$$

$$\int_0^2 f-g - \int_2^4 f-g + \int_4^5 f-g$$

$$\int_0^2 (x^3 - 6x^2 + 8x)\,dx - \int_2^4 (x^3 - 6x^2 + 8x)\,dx + \int_4^5 \ast$$

$$= \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 - \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_2^4 + \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_4^5$$

$$= \left[\frac{2^4}{4} - 2(2)^3 + 4(2)^2 - (0)\right] - \left[\frac{4^4}{4} - 2(4)^3 + 4(4)^2 - \left(\frac{2^4}{4} - 2(2)^3 + 4(2)^2\right)\right]$$

$$+ \left[\frac{5^4}{4} - 2(5)^3 + 4(5)^2 - \left(\frac{4^4}{4} - 2(4)^3 + 4(4)^2\right)\right]$$

$$= \frac{16}{4} - 16 + 16 - \left[64 - 128 + 64 - (4 - 16 + 16)\right]$$

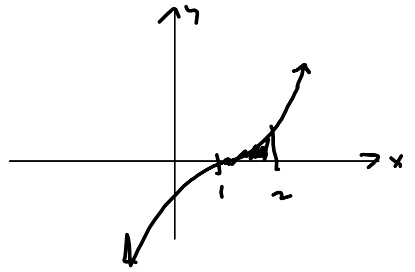
$$+ \left[\frac{5^4}{4} - 250 + 100 - (64 - 128 + 64)\right]$$

$$= 4 - [-64 + 64 - 4] + \frac{5^4}{4} - 150 - 0$$

$$= 8 + \frac{5^4}{4} - 150 = \frac{5^4}{4} - 142 = \frac{625 - 568}{4} = \frac{57}{4}$$

(2) Consider area bdd by $y=(x-1)^3$, $x=1$, $x=2$, and $y=0$

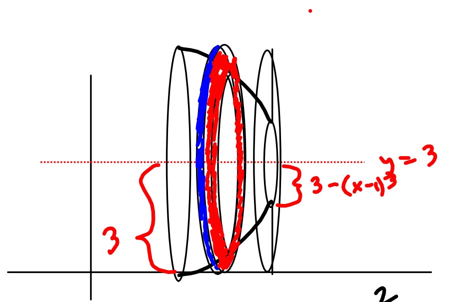
(a) Sketch the region and find its area



Area $= \int_1^2 (x-1)^3\,dx = \frac{(x-1)^4}{4}\Big]_1^2 = \frac{1^4}{4} - \frac{0^4}{4}$

$= \boxed{\frac{1}{4} = \text{Area}}$

(b) Sketch the solid obtained by rotating the region about the line $y=3$. ~~Find its volume~~ Write the integral. Show a representative washer.



Volume $\approx \pi \sum_{k=1}^{n} (\text{outer}^2 - \text{inner}^2)\Delta x$

$\xrightarrow{n\to\infty} \boxed{\pi\int_1^2 \left(3^2 - (3-(x-1)^3\right)^2 dx}$

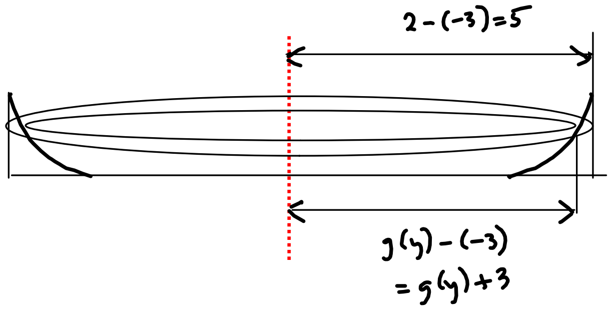
(c) 5pts Evaluate $\pi\int_1^2 \left(3^2 - (3-(x-1)^3)^2\right) dx$

$= \pi\int_1^2 9 - (9 - 2(3)(x-1)^3 + (x-1)^6)\,dx$

$= \pi\int_1^2 (9 - 9 + 6(x-1)^3 - (x-1)^6)\,dx = \pi\int_1^2 (6(x-1)^3 - (x-1)^6)\,dx$

$= \pi\left[\frac{6}{4}(x-1)^4 - \frac{1}{7}(x-1)^7\right]_1^2 = \frac{3\pi}{2} - \frac{1}{7} = \frac{21\pi - 2\pi}{14} = \boxed{\frac{19\pi}{14}}$

(d) 5pts Same region. Rotate about $x=-3$ Sketch & write integral



$y = (x-1)^3 = y$

$x-1 = \sqrt[3]{y}$

$x = \sqrt[3]{y} + 1 = g(y)$

$$\boxed{\text{Volume} = \pi \int_0^1 \left(5^2 - (\sqrt[3]{y}+1)^2\right) dy}$$

$=$

(e) 5pts Eval. Part c

$$= \pi \int_0^1 \left(25 - \left((\sqrt[3]{y})^2 + 2\sqrt[3]{y} + 1\right)\right) dy$$

$$= \pi \int_0^1 \left(25 - (y^{\frac{1}{3}})^2 - 2y^{\frac{1}{3}} - 1\right) dy$$

$$= \pi \int_0^1 \left(24 - y^{\frac{2}{3}} - 2y^{\frac{1}{3}}\right) dy$$

$$= \pi \left[24y - \frac{3}{5}y^{\frac{5}{3}} - 2\left(\frac{3}{4}\right)y^{\frac{4}{3}}\right]_0^1$$

$$= \pi\left[24 - \frac{3}{5} - \frac{3}{2}\right] = \pi\left[\frac{240-6-15}{10}\right] = \pi\left[\frac{219}{10}\right]$$

$$= \boxed{\frac{219\pi}{10}}$$

(3) $f(x) = x^2+5x+11$ is 1-to-1 on $[-\frac{5}{2},\infty)$

(a) (5pts) Find f^{-1}. State its $\mathcal{D}$ & $\mathcal{R}$. Find $f^{-1}(5)$

$x = y^2+5y+11 = x \longrightarrow$

$y^2+5y+\left(\frac{5}{2}\right)^2-\frac{25}{4}-\frac{44}{4} = x$

$\left(y+\frac{5}{2}\right)^2 = x - \frac{}{4}$

$y+\frac{5}{2} = \pm\sqrt{\frac{4x-19}{4}} = \pm\frac{\sqrt{4x-9}}{2}$

$y = \frac{-5\pm\sqrt{4x-19}}{2} \longrightarrow \boxed{f^{-1}(x) = \frac{-5+\sqrt{4x-19}}{2}}$

From $\mathcal{D}(f) = [-\frac{5}{2},\infty)$

& $\mathcal{R}(f) = \mathcal{D}(f^{-1}) = \{x \mid 4x-19 \geq 0\} = \{x \mid x \geq \frac{19}{4}\} = \boxed{[\frac{19}{4},\infty) = \mathcal{D}(f^{-1})}$

$\longrightarrow \mathcal{R}(f^{-1}) = \mathcal{D}(f) = \boxed{[-\frac{5}{2},\infty) = \mathcal{R}(f^{-1})}$

$f^{-1}(5) = \frac{-5+\sqrt{4(5)-19}}{2} = \frac{-5+\sqrt{20-19}}{2} = \boxed{\frac{-5+1}{2} = \boxed{f^{-1}(5) = 3}}$

Check $x^2+5x+11 = 5$

$x^2+5x = 5-11 = -6$

$x^2+5x+\left(\frac{5}{2}\right)^2 = -\frac{6}{1}\cdot\frac{4}{4}+\frac{25}{4} = \frac{1}{4} \longrightarrow$

$(x+5/2)^2 = \frac{1}{4}$

$x+\frac{5}{2} = \pm\frac{1}{2}$

$x = \frac{-5\pm1}{2}$

$\nearrow \frac{-5+1}{2} = -\frac{4}{2} = \boxed{-2 = f^{-1}(5)}$

$\searrow \frac{-5-1}{2} = -\frac{6}{2} = -3 < -\frac{5}{2} \notin \mathcal{D}(f)$

(b) (5pts) Find $(f^{-1})'(x)$

$f^{-1}(x) = \frac{-5+\sqrt{4x-19}}{2} = -\frac{5}{2}+\frac{1}{2}(4x-19)^{\frac{1}{2}}$

$\longrightarrow (f^{-1})'(x) = \frac{1}{4}(4x-19)^{-\frac{1}{2}}(4) = (4x-19)^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{4x-19}} = (f^{-1})'(x)$

$\longrightarrow (f^{-1})'(5) = (20-19)^{-\frac{1}{2}} = \boxed{1 = (f^{-1})'(5)}$

(c) (5pts) Find $(f^{-1})'(5)$ using a theorem.

$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(5))}$

$f'(x) = 2x+5$

$(f^{-1})(5) = -2 \longrightarrow (f^{-1})'(5) = \frac{1}{2(-2)+5} = \frac{1}{-4+5} = 1 = (f^{-1})'(5)$

4) Differentiate the following.

a) 5 pts $y = 7 \cdot 6^{x^2-3x} \rightarrow \boxed{y' = 7\ln(6)\left(6^{x^2-3x}\right)(2x-3)}$

b) 5 pts $y = \ln\left(\dfrac{\sqrt[5]{x^2-3x}}{(3x^5+5x)^3}\right) = \ln\left(\dfrac{(x^2-3x)^{\frac{1}{5}}}{(3x^5+5x)^3}\right)$

$= \frac{1}{5}\ln(x^2-3x) - 3\ln(3x^3+5x) \rightarrow$

$\boxed{y' = \frac{1}{5}\left(\dfrac{2x-3}{x^2-3x}\right) - 3\left(\dfrac{15x^2+5}{3x^2+5x}\right)}$

c) 5 pts $y = [\sin(x)]^{x^2-3x} \rightarrow$

$\ln(y) = (x^2-3x)\ln(\sin(x)) \rightarrow$

$\dfrac{y'}{y} = (2x-3)\ln(\sin(x)) + (x^2-3x)\left(\dfrac{\cos(x)}{\sin(x)}\right)$

$\rightarrow \boxed{y' = (2x-3)\ln(\sin(x)) + (x^2-3x)(\cot(x))\left(\sin(x)\right)^{x^2-3x}}$

d) 5 pts $y = (x^4-7x^2)^5(\sin^8(x)) \rightarrow$

$\ln(y) = 5\ln(x^4-7x^2) + 8\ln(\sin(x)) \rightarrow$

$\dfrac{y'}{y} = 5\left(\dfrac{4x^3-14x}{x^4-7x^2}\right) + 8\left(\dfrac{\cos(x)}{\sin(x)}\right)$

$\Rightarrow \boxed{y' = \left[5\left(\dfrac{4x^3-14x}{x^4-7x^2}\right) + 8\left(\cot(x)\right)\right](x^4-7x^2)^5(\sin^8(x))}$

e 5pts $y = \ln(x^2-3x) \longrightarrow \boxed{y' = \dfrac{2x-3}{x^2-3x}}$

f 5pts $y = \log_7(x^2-3x) \; \boxed{= \dfrac{1}{\ln(7)}\left(\dfrac{2x-3}{x^2-3x}\right) = y'}$

5 BONUS 5pts $\int \cot(x)\,dx = \int \dfrac{\cos(x)}{\sin(x)}\,dx = \int \dfrac{du}{u} = \ln|u| + C$

$= \boxed{\ln|\sin(x)| + C}$

$u = \sin(x)$

$du = \cos(x)\,dx$