2410 WEEK 6 SOLNS MILLS

1. We differentiate the following wrt x.

(a) 5pts $f(x)=\sin(x^3) \Rightarrow \boxed{f'(x)=3x^2\cos(x^3)}$

(b) 5pts $f(x)=\sin^3(x) \Rightarrow \boxed{f'(x)=3\sin^2(x)\cos(x)}$

(c) 5pts $f(x)=\sin^3(x^3) \Rightarrow \boxed{f'(x)=3\sin^2(x^3)\cos(x^3)\cdot 3x^2}$

(d) 5pts $f(x)=\dfrac{\sec^5(x)}{\sqrt{x^2\sin^3(x)+1}} \Rightarrow$

$$f'(x)=\frac{5\sec^4(x)\sec(x)\tan(x)\sqrt{x^2\sin^3(x+1)}}{x^2\sin^3(x)+1}$$

HOLY SMOKES!

$$-\frac{\sec^5(x)\left(\frac{1}{2}\right)\left(x^2\sin^3(x)+1\right)^{-\frac{1}{2}}\left(2x\sin^3(x)+x^2(3\sin^2(x)\cos(x))\right)}{x^2\sin^3(x)+1}$$

2410 MILLS

2) We find $y' = \frac{dy}{dx}$ for the equation $(x-2)^2 + (y+7)^2 = 25$ in two ways:

a) 5pts we solve for y & differentiate explicitly. We are looking at the bottom half of a circle.

$$(y+7)^2 = 25-(x-2)^2$$
$$\Rightarrow y+7 = \pm\sqrt{25-(x-2)^2} \Rightarrow \boxed{y = -7-\sqrt{25-(x-2)^2} = f(x)}$$

since we want the bottom half.

$$y = -7-(25-(x-2)^2)^{\frac{1}{2}} \Rightarrow$$
$$\boxed{y' = f'(x) = -\frac{1}{2}(25-(x-2)^2)^{-\frac{1}{2}}(-2(x-2))} = \frac{x-2}{\sqrt{25-(x-2)^2}}$$

b) 5pts with implicit differentiation:

$$(x-2)^2 + (y+7)^2 = 25 \Rightarrow$$
$$2(x-2) + 2(y+7)y' = 0 \Rightarrow$$
$$(x-2) + (y+7)y' = 0 \Rightarrow$$
$$(y+7)y' = -x+2 \Rightarrow$$
$$y' = \frac{-x+2}{y+7} \text{ or } \boxed{-\frac{(x-2)}{y+7} = y'}$$

(c) (5 pts) We find an eq'n of the tangent line to the circle at the point $(-2,-10)$.

m1

$$y'(-2) = -\frac{1}{2}\left(25-(-2-2)^2\right)^{-\frac{1}{2}}\left(-2(-2-2)\right)$$

$$= -\frac{1}{2}\left(25-(4^2)\right)^{-\frac{1}{2}}\left(-2(-4)\right)$$

$$= -\frac{1}{2}(9)^{-\frac{1}{2}}(8)$$

$$= -4\left(\frac{1}{\sqrt{9}}\right) = \boxed{-\frac{4}{3} = y'(-2) = m = f'(-2)}$$

$$y = m(x-x_1)+y_1 = f'(x_1)(x-x_1)+f(x_1)$$

$$= f'(-2)(x-(-2)) + f(-2)$$

$$= -\frac{4}{3}(x+2) + f(-2)$$

$$= \boxed{-\frac{4}{3}(x+2)-10 = L(x) = \text{Tangent Line}}$$

scratch

$$f(-2) = -7-\sqrt{25-(-2-2)^2}$$
$$= -7-\sqrt{25-4^2}$$
$$= -7-\sqrt{25-16}$$
$$= -7-\sqrt{9} =$$
$$= -7-3 = -10$$

m2:

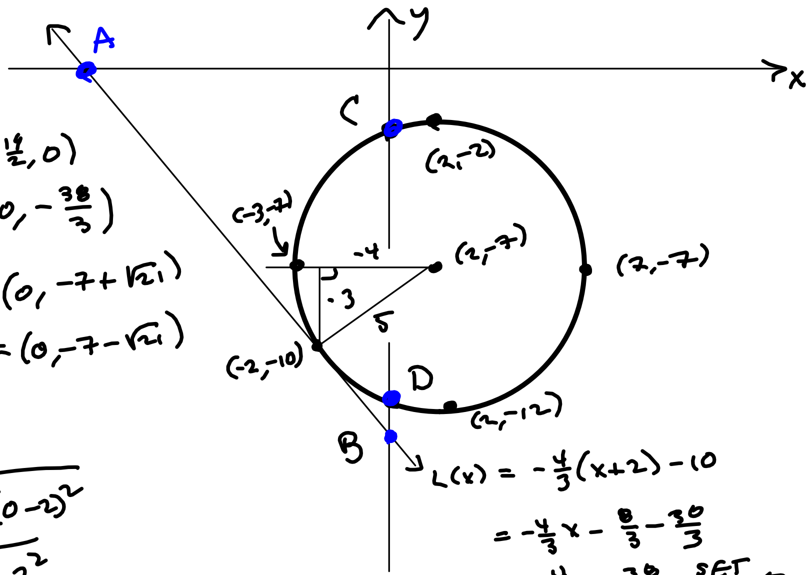
$$-\frac{(x-2)}{y+7} = y'\Big|_{(x,y)=(-2,-10)} = m_{tan} = -\frac{-2-2}{-10+7} = -\frac{-4}{-3} = -\frac{4}{3} = m$$

$$y = m(x-x_1)+y_1$$

$$\boxed{y = -\frac{4}{3}(x+2)-10}$$

(d) (5pts) we sketch all of the above

Rough Sketch.



$A = \left(-\frac{19}{2}, 0\right)$

$B = \left(0, -\frac{38}{3}\right)$

$C = \left(0, -7+\sqrt{21}\right)$

$D = \left(0, -7-\sqrt{21}\right)$

Find C & D:

$x=0 \Rightarrow$

$$y = -7 \pm \sqrt{25-(0-2)^2}$$
$$= -7 \pm \sqrt{25-2^2}$$
$$= -7 \pm \sqrt{21}$$

$$L(x) = -\frac{4}{3}(x+2) - 10$$
$$= -\frac{4}{3}x - \frac{8}{3} - \frac{30}{3}$$
$$= -\frac{4}{3}x - \frac{38}{3} \overset{\text{SET}}{=} 0$$

$$\Rightarrow -\frac{4}{3}x = \frac{38}{3}$$
$$\Rightarrow -4x = 38$$
$$\Rightarrow x = -\frac{38}{4} = -\frac{19}{2}$$

2410 MILLS

3 Use implicit differentiation to find $y' = \frac{dy}{dx}$

a 5pts

$$x^2+4xy+y^2=4 \Longrightarrow$$

$$2x+4y+4xy'+2yy'=0 \Longrightarrow$$

$$(4x+2y)y' = -2x-4y \Longrightarrow$$

$$y' = \frac{-2x-4y}{4x+2y} = \boxed{\frac{-x-2y}{2x+y} = y'}$$

b 5pts

$$xy\sin(xy)+y^2=5 \Longrightarrow$$

$$y\sin(xy)+xy'\sin(xy)+xy\cos(xy)(y+xy') = 0$$

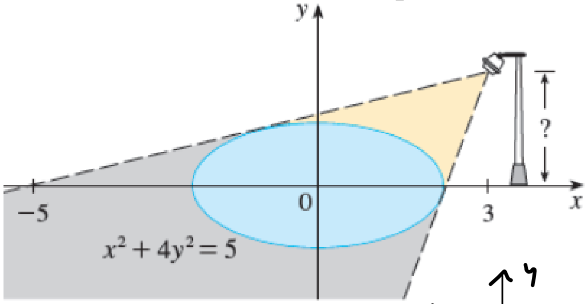
$$\Rightarrow x\sin(xy)y' + xy^2\cos(xy) + x^2y\cos(xy)y' = -y\sin(xy)$$

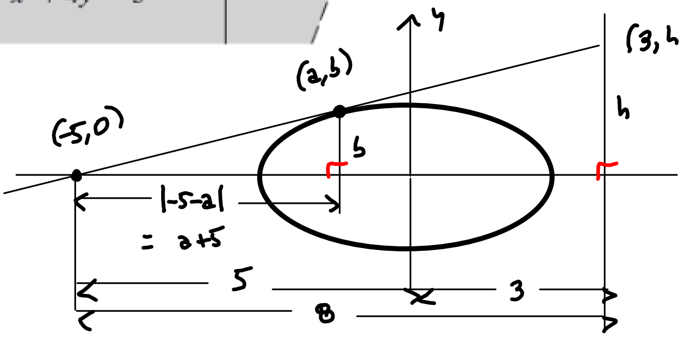
$$\Rightarrow \left(x\sin(xy)+x^2y\cos(xy)\right)y' = -y\sin(xy) - xy^2\cos(xy)$$

$$\Longrightarrow \boxed{y' = \frac{-y\sin(xy)-xy^2\cos(xy)}{x\sin(xy)+x^2y\cos(xy)}}$$

2410 MILLS

4. **(10 pts) The figure shows a lamp located 3 units to the right of the *y*-axis and a shadow created by the elliptical region $x^2 + 4y^2 \le 5$. If the point (-5, 0) is on the edge of the shadow, how far above the *x*-axis is the lamp located?**





$a < 0$ so distance from $(-5,0)$ to $(a,0)$ is $|-5-a| = |-(5+a)| = |a+5| = a+5$. Looks weird, but $a<0$, and $|a|<5$ means $|a+5| = a+5 > 0$

$x^2+4y^2=5$

$2x+8yy'=0$

$y' = -\frac{2x}{8y} = -\frac{x}{4y}$. using $(x,y)=(a,b)$ & similar triangles

$\frac{h}{8} = m_{tan} = \frac{-2x}{8y} = \frac{-x}{4y} = \frac{-a}{4b} = \frac{b}{5+a} = \frac{b}{a+5}$

$\Rightarrow -5a-a^2 = 4b^2 \Rightarrow$

$a^2+4b^2 = -5a$

This is the left-hand side of the ellipse's equation.

(a,b) on the ellipse $\Rightarrow$

$a^2+4b^2=5 \Rightarrow -5a=5 \Rightarrow \boxed{a=-1}$

$\Rightarrow a^2+4b^2 = 1^2+4b^2 = 4b^2+1 = 5$

$\Rightarrow 4b^2=4 \Rightarrow b^2=1 \Rightarrow b=\pm 1$

$\Rightarrow \boxed{b=1}$, by picture.

So, $(a,b)=(-1,1) \Rightarrow \frac{h}{8} = \frac{b}{a+5} = \frac{1}{-1+5}$

$= \frac{1}{4}$

$\Rightarrow h = (\frac{1}{4})(8) = \boxed{2=h}$

$x^2y^2 - 2xy^3 = \cos(xy)$

$2xy^2 - x^2(2yy') - 2y^3 - 2x(3y^2y') = (-\sin(xy))(y + xy')$

$= -y\sin(xy) - x\sin(xy)y'$

$-2x^2yy' - 2x(3y^2)y' + x\sin(xy)y' = -y\sin(xy) - 2xy^2 + 2y^3$

$$\boxed{y' = \frac{-y\sin(xy) - 2xy^2 + 2y^3}{-2x^2y - 6xy^2 + x\sin(xy)}}$$