2410 Week 8 PRACTICE Solutions Mills

1. Let. $f(x) = 2x^3 - 6x^2 - 90x$ $\longrightarrow$ $f'(x) = 6x^2 - 12x - 90$

   a. (5 pts) Convince me that there is a point $c \in [1,10]$ such that $f'(c)$ is the same as the average slope, m_{avg}, of f on the interval $[1,10]$, *without finding c, itself!*

f is cnt^s and difb^{le} everywhere, b/c f is a polynomial.

$\therefore$ f is cnt^s on $[1,10]$ and difb^{le} on $(1,10)$.

$\therefore \exists\, c \in (1,10) \ni f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(10)-f(1)}{10-1} = m_{avg}$, by MVT.

   b. (5 pts) What is the average slope, m_{avg}, of f on the interval $[0,3]$? What is $f'(x)$? Find c.

$f(x) = 2x^3 - 6x^2 - 90x \longrightarrow$ $\boxed{f'(x) = 6x^2 - 12x - 90}$

$m_{avg} = \frac{f(10)-f(1)}{10-1} = \frac{500-(-94)}{9} = \frac{594}{9} = \boxed{66 = m_{AVG}}$

Scratch:

Synthetic Division is the most efficient way to evaluate this polynomial by hand, by the Remainder Theorem. I flubbed this, live, but I knew I had the m_{avg} right on the Maple.

$$\begin{array}{r|rrrr} 10 & 2 & -6 & -90 & 0 \\ & & 20 & 140 & 500 \\ \hline & 2 & 14 & 50 & 500 \end{array}$$

was -14. $\to f(10)$

$$\begin{array}{r|rrrr} 1 & 2 & -6 & -90 & 0 \\ & & 2 & -4 & -94 \\ \hline & 2 & -4 & -94 & -94 \end{array}$$

$\to f(1)$

Looks like MVT

Find c.

$f'(x) = 6x^2 - 12x - 90 \overset{SET}{=} 66 = m_{avg}$ $\longrightarrow$

$6x^2 - 12x - 156 = 0$ This is $g(x) = 0$ $\longrightarrow$

$\longrightarrow x^2 - 2x - 26 = 0$

$\longrightarrow x^2 - 2x + 1^2 - 1 - 26 = (x-1)^2 - 27 = 0$

$\longrightarrow x = 1 \pm \sqrt{27} = 1 \pm 3\sqrt{3}$, and $\boxed{c = 1 + 3\sqrt{3}} \in (1,10)$, as desired.

Desmos implementation:

$f'(x) = m_{avg}$ $\longrightarrow$

$g(x) = f'(x) - m_{avg} = 0$

Confirming this is a little tough, but Desmos does a fair job.

Define $f(x)$ and $f'(x)$.

Calculate m_{avg}

Solve $f'(x) = m_{avg}$, by creating

$g(x) = f'(x) - m_{avg}$, and find find any intercepts in $[1,10]$.

The sol'n to this last eq'n should match your $1+3\sqrt{3}$.

$f(1+3\sqrt{3})$ is horribly painful to do by hand.

Good practice for algebra & arithmetic chops.

24.0

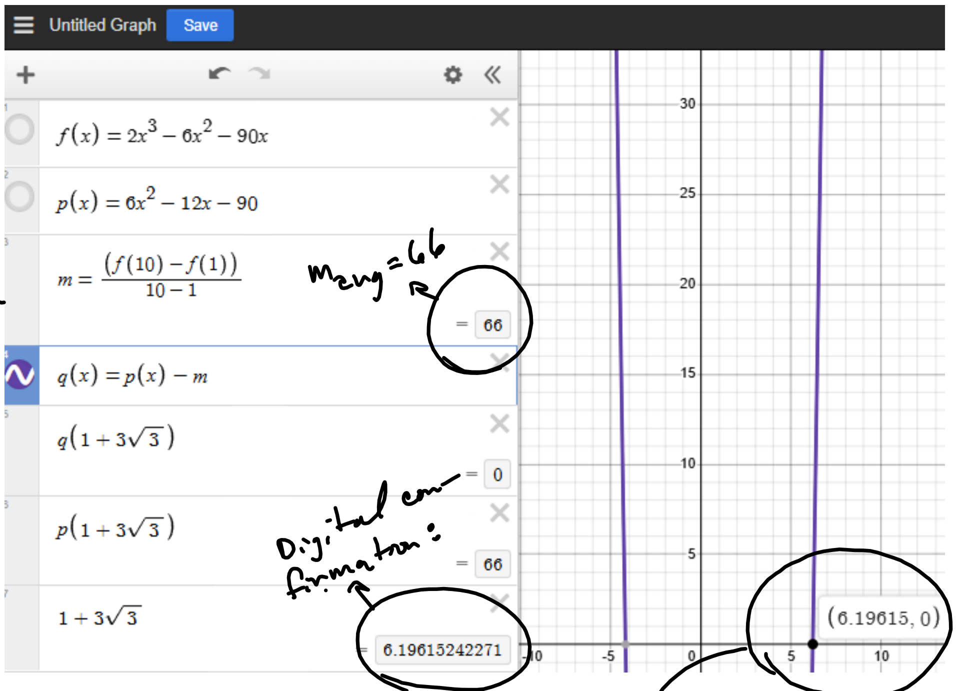
Mills

Desmos will get you a digital solution and it will confirm an exact solution, but the free graphing calculator won't give you an *exact* answer. That takes a CAS. I think wolframalpha.com's free version is pretty powerful, if you leanr how to talk to it.

$p(x) = f'(x)$

$g(x) = f'(x) - m_{avg}$

All turned off



$c \approx 6.19615$

24w Mills

2. Let $f(x) = (x+5)^3 (x-6)^2$.

a. (5 pts) Find the absolute maximum and minimum of f on the interval [0, 3].

Check the endpoints:

$f(0) = 5^3(-6)^2 = 125(36) = 4500$

$f(3) = \overset{g}{(3+5)^3}\overset{h}{(3-6)^2} = 8^3(-3)^2 = 9(8)^3(64) = 9(512) = 4608$

$f'(x) = \overbrace{3(x+5)^2(1)}^{g'}\overbrace{(x-6)^2}^{h} + \overbrace{(x+5)^3}^{g}\overbrace{2(x-6)^1(1)}^{h'}$

$= 3(x+5)^2(x-6)^2 + 2(x+5)^3(x-6)$ SET 0

$\Rightarrow (x+5)^2(x-6)\left(3(x-6) + 2(x+5)\right)$

$= (x+5)^2(x-6)(3x-18+2x+10)$

$= (x+5)^2(x-6)(5x-8) = 0$

$\Rightarrow x \in \{-5, \frac{8}{5}, 6\}$

The only critical # in $[0,3]$

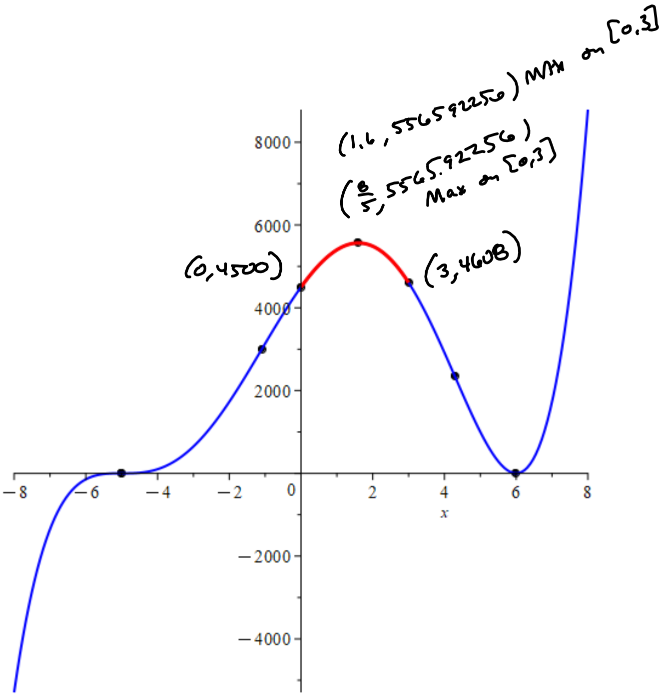
$f\left(\frac{8}{5}\right) = \left(\frac{8}{5}+5\right)^3\left(\frac{8}{5}-6\right)^2 = \left(\frac{8+25}{5}\right)^3\left(\frac{8-30}{5}\right)^2 = \left(\frac{33}{5}\right)^3\left(-\frac{22}{5}\right)^2$

$= \frac{17393508}{3125} = 5565.922560$

(This happens to be exact.)

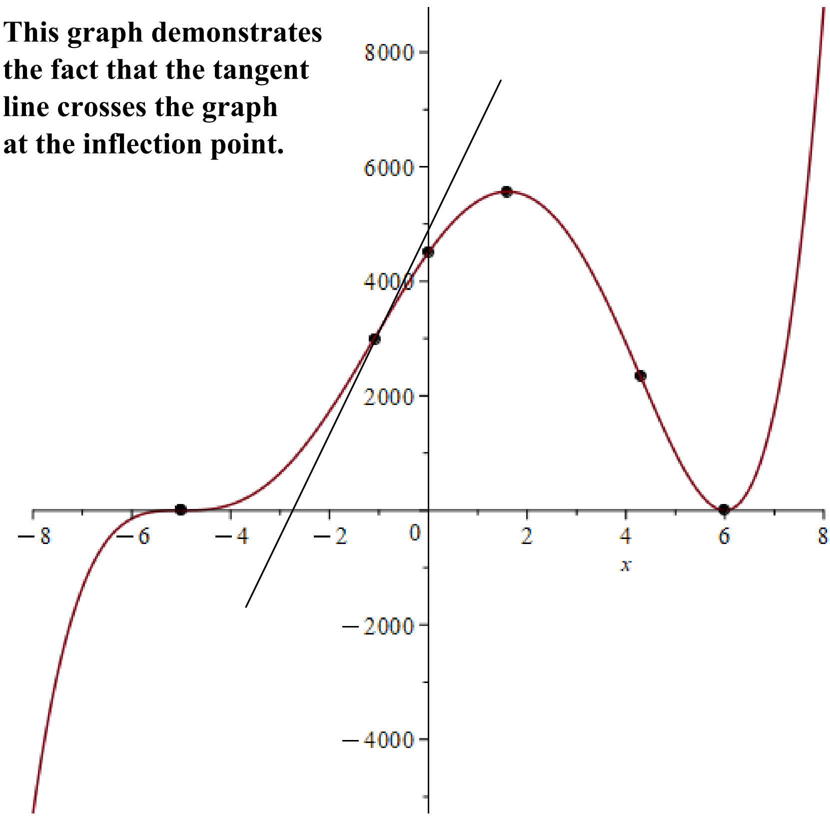
$4500 = f(0)$
$4608 = f(3)$
$5565.92256 = f\left(\frac{8}{5}\right)$

$$\boxed{\begin{array}{l} f\left(\frac{8}{5}\right) = 5565.92256 \text{ is Max} \\ f(0) = 4500 = \text{MIN} \end{array}}$$



24.0 Mills

This graph demonstrates the fact that the tangent line crosses the graph at the inflection point.

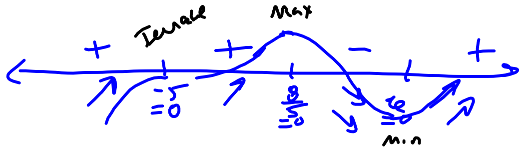


2410 Mills

b. (5 pts) Find the *open* intervals on which f is increasing. Find the open intervals on which f is decreasing.

$f'(x)=0 \Rightarrow x \in \{-5, \frac{8}{5}, 6\}$ & this is ALL of the critical #s

$(x+5)^2(5x-8)'(x-6)'$ is of degree 4



Increasing: $(-\infty, \frac{8}{5}) \cup (6, \infty)$ WebAssign accepts $(-\infty, \frac{8}{5})$, $(6, \infty)$, but I don't

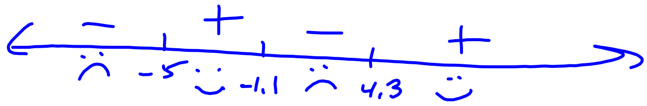
Decreasing: $(\frac{8}{5}, 6)$

c. (5 pts) Find the open intervals on which f is concave up. Find the open intervals on which f is concave down.

$f''(x)=0$ @ $x=-5$ & $\frac{16 \pm 11\sqrt{6}}{10}$ ≈ 4.3, ≈ -1.1

$f''(x) = 2(x+5)(10x^2-32x-47)$

$-1.1, -4.3, -5$



Concave up: $(-5, \frac{16-11\sqrt{6}}{10}) \cup (\frac{16+11\sqrt{6}}{10}, \infty)$

Concave down: $(-\infty, -5) \cup (\frac{16-11\sqrt{6}}{10}, \frac{16+11\sqrt{6}}{10})$

Inflection points @ $x=-5, \frac{16 \pm 11\sqrt{6}}{10}$

$f''(x) = 2(x+5)(10x^2-32x-47) \overset{SET}{=} 0$ x^3

$\Rightarrow x+5=0$ or $10x^2-32x-47=0$

$x=-5$ $\Rightarrow 10(x^2 - \frac{32}{10}x - \frac{47}{10}) = 0$

$3 \overline{)363}$ $11 \overline{)121}$ 11

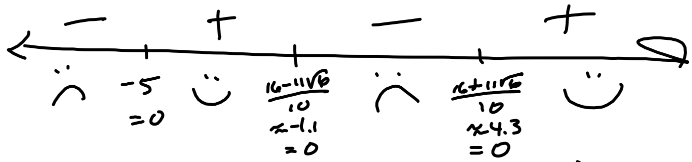
$\Rightarrow x^2 - \frac{16}{5}x - \frac{47}{10} = x^2 - \frac{16}{5}x + (\frac{8}{5})^2 - \frac{64}{25}\cdot\frac{2}{2} - \frac{47}{10}\cdot\frac{5}{5}$

$= (x-\frac{8}{5})^2 + \frac{-128-235}{50} = (x-\frac{8}{5})^2 - \frac{363}{50} = 0$

$\Rightarrow x = \frac{8}{5} \pm \sqrt{\frac{363}{50}} = \frac{8}{5} \pm \frac{11\sqrt{3}}{5\sqrt{2}} = \frac{8}{5} \pm \frac{11\sqrt{6}}{10}$

$x = \frac{16 \pm 11\sqrt{6}}{10}$ → 4.294438717, -1.094438717

$f''=0$ @ $x=-5, -1.1, 4.3$



$f''(x) = 20(x+5)(x-(\frac{16-11\sqrt{6}}{10}))(x-(\frac{16+11\sqrt{6}}{10}))$

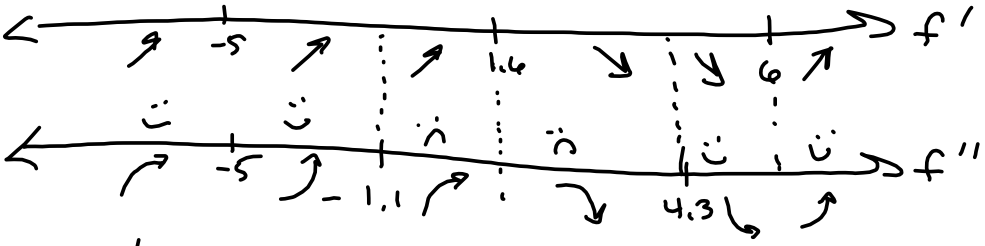
$\boxed{\text{Concave Up } (-5, \frac{16-11\sqrt{6}}{10}) \cup (\frac{16+11\sqrt{6}}{10}, \infty)}$

$\boxed{\text{Concave down}: (-\infty, -5) \cup (\frac{16-11\sqrt{6}}{10}, \frac{16+11\sqrt{6}}{10})}$

24.J MILLS

$f' = 0$ @ $-5, \frac{8}{5}, 6 = -5, 1.6, 6$

$f'' = 0$ @ $-5, -1.1, 4.3$

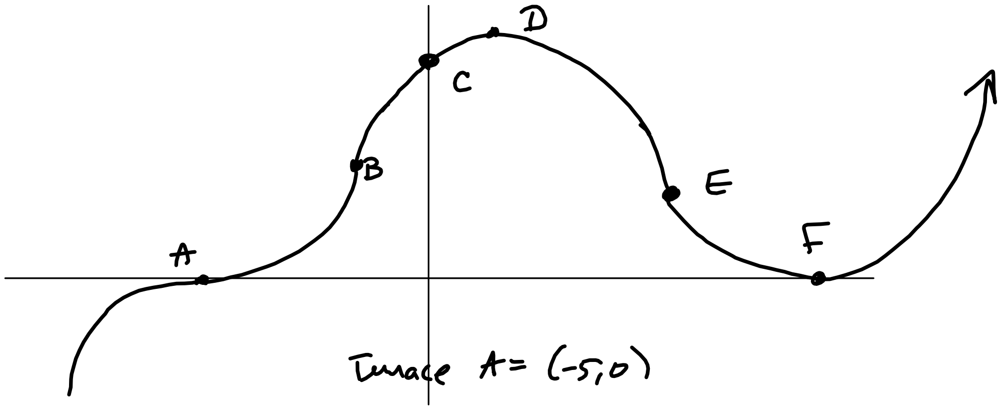


Desmos could do f(x) quant'ta'ively very quickly.

Actual values of

$f(-5), f\left(\frac{16-11\sqrt{6}}{10}\right), f\left(\frac{8}{5}\right), f\left(\frac{16+11\sqrt{6}}{10}\right), f(6)$

to really sharpen it up.



Terrace A = (-5, 0)

I.P. B ≈ (-1.09443871706, 2998.37848895)

y-int C = (0, 4500)

MAX D = (1.6, 5565.92256)

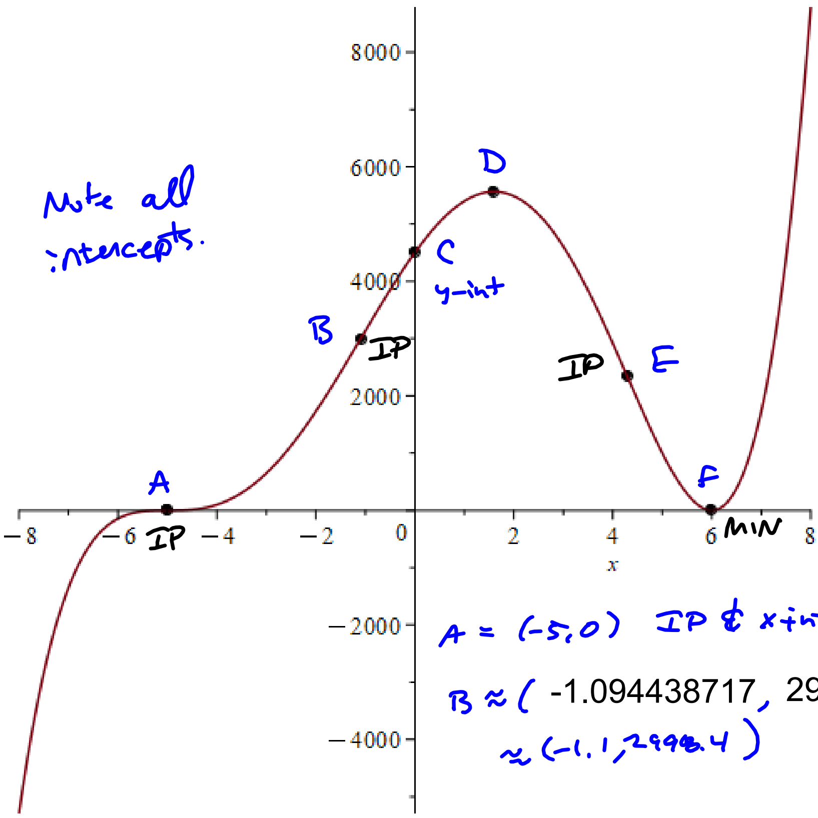
IP E = (4.29443871706, 2335.63063105)

x-int, MIN F = (6, 0)

2410 M1-LS

d. (5 pts) Use all the information from parts a – d to sketch the graph of f. Label all intercepts, max/min points, and inflection points. You may put the ordered-pair labels directly on the graph or make a legend/key as I will demonstrate in lecture.

Note all intercepts.



$A = (-5, 0)$ IP & x-int

$B \approx (-1.094438717, 2998.378489)$ I.P.

$\approx (-1.1, 2998.4)$

2440 MILLS

3. Let $f(x)=(x+2)^2\sqrt{16-x^2}$.

a. (5 pts) What is the domain of f?

Need $16-x^2 \geq 0$

$(4-x)(4+x) \geq 0$

$-x^2$

No — yes + yes — No; -4 $=0$, 4 $=0$

≥ 0

$\rightarrow x \in \boxed{[-4,4] = D(f)}$

b. (5 pts) Use a graphing utility to sketch the graph of f. Include all max/min values and intercepts. Round answers to 2 decimal places.

$\approx (-3.62, 4.47)$

$x(-2,0)$

$= (0,16)$

$\approx (2.95, 66.19)$

Desmos Work

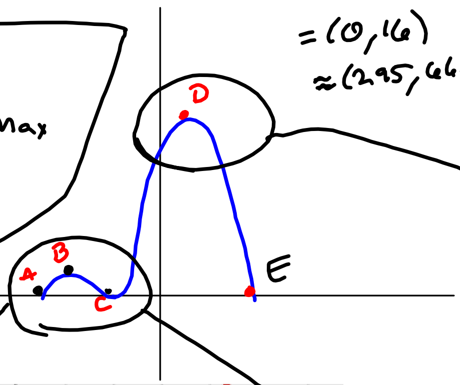
$A = (-4,0)$ min

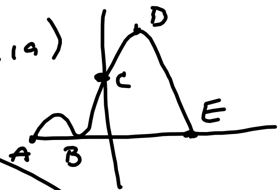
$B \approx (-3.62, 4.47)$ Max

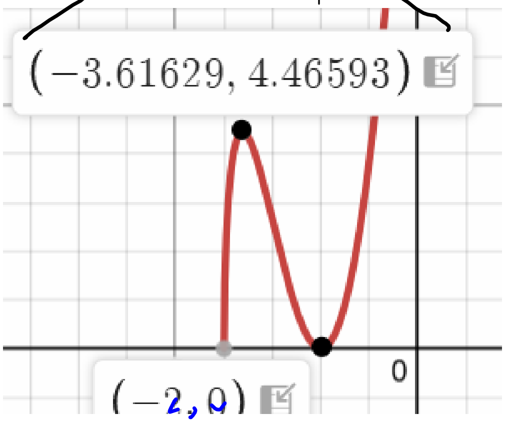
$C = (-2,0)$ min

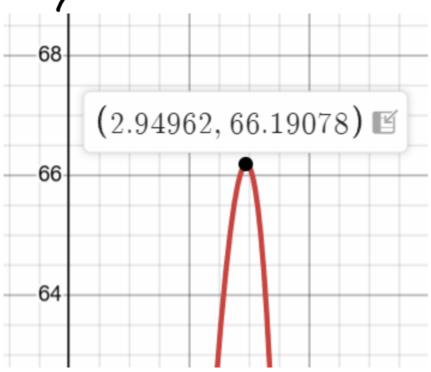
$D \approx (2.95, 66.19)$

$E = (4,0)$ MIN









2410 M.UCS

c. (**Bonus 5 pts**) Use calculus to find the *exact* maximum value. What is the range of f?

$(x+2)^2\sqrt{16-x^2} = f(x) = (x+2)^2(16-x^2)^{\frac{1}{2}}$

$\longrightarrow f'(x) = 2(x+2)'(1)(16-x^2)^{\frac{1}{2}} + (x+2)^2\left(\frac{1}{2}(16-x^2)^{-\frac{1}{2}}(-2x)\right)$

$= \frac{2(x+2)(16-x^2)^{\frac{1}{2}}}{1} \cdot \frac{(16-x^2)^{\frac{1}{2}}}{(16-x^2)^{\frac{1}{2}}} - \frac{x(x+2)^2}{(16-x^2)^{\frac{1}{2}}}$

$= \frac{2(x+2)(16-x^2) - x(x+2)^2}{(16-x^2)^{\frac{1}{2}}} = \frac{(x+2)\left(2(16-x^2) - x(x+2)\right)}{\text{same}}$

$= \frac{(x+2)(32-2x^2-x^2-2x)}{\text{same}} = \frac{(x+2)(-3x^2-2x+32)}{(16-x^2)^{\frac{1}{2}}} \overset{\text{SET}}{=} 0$

2410 MILLS

$\Rightarrow x+2=0$ or $-3x^2-2x+32=0$

$\Rightarrow x=-2$ $\Rightarrow x^2+\frac{2}{3}x-\frac{32}{3}=0$

$\Rightarrow x^2+\frac{2}{3}x+\left(\frac{1}{3}\right)^2=\frac{32}{3}\cdot\frac{3}{3}+\frac{1}{9}=\frac{96+1}{9}$

$\Rightarrow \left(x+\frac{1}{3}\right)^2=\frac{97}{9} \Rightarrow$

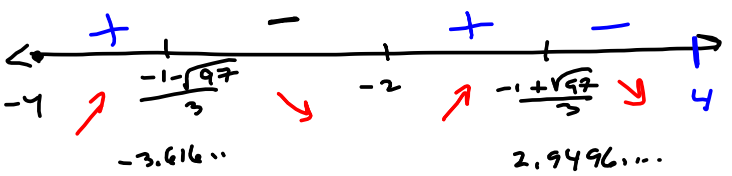
$\Rightarrow x=\frac{-1\pm\sqrt{97}}{3}$

Critical #s:

$x=-2,\ \frac{-1-\sqrt{97}}{3},\ \frac{-1+\sqrt{97}}{3}$ → the one on the right

By our work $f'(a)=\dfrac{-3(x+2)^1\left(x-\left(\frac{-1-\sqrt{97}}{3}\right)\right)^1\left(x-\left(\frac{-1+\sqrt{97}}{3}\right)\right)^1}{\sqrt{16-x^2}}$

from test-value $x=-3$



If you can "see" the factored form, you only need one test value & logic!

Intervals

$(-4, -3.6\ldots)$

$(-3.6, -2)$ $\quad -3 \quad f'(-3)=\dfrac{(-3+2)(-3(-3)^2-2(-3)+32)}{\sqrt{-}}=\dfrac{(-)(-27+6+32)}{+}$

$(-2, 2.9)$ $\quad = -$

-4.1576092031

2410 MILLS

You can do some things with Wolframalpha.com, as well.

https://www.wolframalpha.com/

Enter either of the following commands:

Differentiate (x+2)^2*sqrt(16-x^2)

diff((x+2)^2*sqrt(16-x^2),x)

Wolfram Alpha will spit out the derivative:

$$\frac{d}{dx}\left((x+2)^2\sqrt{16-x^2}\right) = -\frac{(x+2)\left(3x^2+2x-32\right)}{\sqrt{16-x^2}}$$

Scroll down the page and see an Alternate Form that you can copy and paste back into the input window:

Alternate form

$$-\frac{(x+2)\left(x(3x+2)-32\right)}{\sqrt{16-x^2}}$$

What you paste will look like this in the input window:

-((x + 2) (x (3 x + 2) - 32))/sqrt(16 - x^2)

This is an expression you can solve for zero or differentiate it yet again.

Differentiate -((x + 2) (x (3 x + 2) - 32))/sqrt(16 - x^2)

returns the following:

$$\frac{2\left(3x^4+4x^3-72x^2-96x+224\right)}{\left(16-x^2\right)^{3/2}}$$

You can paste THIS into the input window, Put a solve in front of it and an =0 after it!

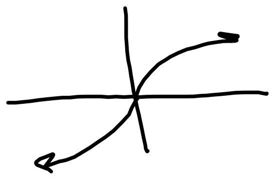
solve (2 (x (3 x + 4) (x^2 - 24) + 224))/(16 - x^2)^(3/2) =0

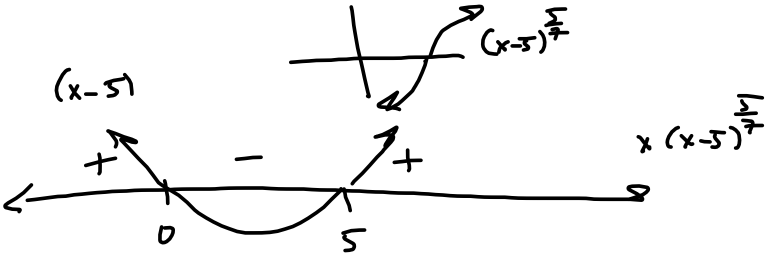
2410 MILLS

4. (5 pts) Let $f(x) = x(x-5)^{\frac{5}{7}}$. Sketch the graph of f. Clearly label all x- and y-intercepts, local max/min points, and inflection points. Each label should be an ordered pair or a letter referring to an ordered pair in a key or legend for the sketch. It's vital that your sketch capture the main features and shape.

$(x-5)^{\frac{5}{7}}$ is in the $f(x) = x^{\frac{1}{3}}$

$0 < \frac{odd_1}{odd_2} < 1$





$f'(x) = (x-5)^{\frac{5}{7}} + x\left(\frac{5}{7}(x-5)^{-\frac{2}{7}}\right)$

$= \frac{(x-5)^{\frac{5}{7}}}{1} \cdot \frac{7(x-5)^{\frac{2}{7}}}{7(x-5)^{\frac{2}{7}}} + \frac{5x}{7(x-5)^{2/7}} = \frac{7(x-5)+5x}{7(x-5)^{\frac{2}{7}}}$

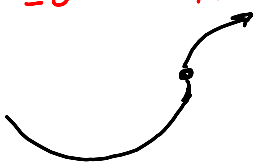
$= \frac{12x-35}{7(x-5)^{2/7}}$

critical #s $x = \frac{35}{12}, 5 = 2 + \frac{11}{12}, 5 = \boxed{2.91\overline{6}, 5 = x}$

− + + f'

$\frac{35}{12}$ 5

$=0$ $\cancel{\nexists}$



$\frac{d}{dx}[$

$-\frac{9}{7})$

2410 MILLS

Heres the mistake

$7(x-5)^{\frac{2}{7}}]$

$$f''(x) = \left(\frac{g}{h}\right)' = \frac{g'h - gh'}{h^2} = \frac{12\left(7(x-5)^{\frac{2}{7}}\right) - (12x-35)\left(\frac{2}{7}\right)\left(7(x-5)\right)}{49(x-5)^{\frac{4}{7}}}$$

$$= \frac{1}{49(x-5)^{\frac{4}{7}}}\left[\frac{12\left(7(x-5)^{\frac{2}{7}}\right)}{1}\cdot\frac{(x-5)^{\frac{9}{2}}}{(x-5)^{9/7}} - \frac{2(12x-35)}{(x-5)^{9/7}}\right]$$

$$= \frac{1}{49(x-5)^{\frac{4}{7}}}\left[\frac{84(x-5)^{\frac{11}{7}} - (24x-70)}{(x-5)^{9/7}}\right]$$

$$f'(x) = \frac{12x-35}{7(x-5)^{2/7}} = \frac{g}{h} \longrightarrow$$

$$f''(x) = \left(\frac{g}{h}\right)' = \frac{g'h - gh'}{h^2} = \frac{12\left(7(x-5)^{\frac{2}{7}}\right) - (12x-35)\left(7\left(\frac{2}{7}\right)(x-5)^{-\frac{5}{7}}\right)}{49(x-5)^{4/7}}$$

$$= (x-5)^{-5/7}\left[\frac{12\left(7(x-5)\right) - 2(12x-35)}{49(x-5)^{4/7}}\right]$$

$$= \frac{84x - 420 - 24x + 70}{49(x-5)^{9/7}} = \frac{60x-350}{49(x-5)^{9/7}} = \frac{10(6x-35)}{49(x-5)^{9/7}}$$

2410 MILLS

$f'(x) = \frac{12x-35}{7(x-5)^{2/7}}$ Lets try f'' again using Product Rule.

$$= (12x-35)\left(\frac{1}{7}(x-5)^{-\frac{2}{7}}\right) \longrightarrow$$

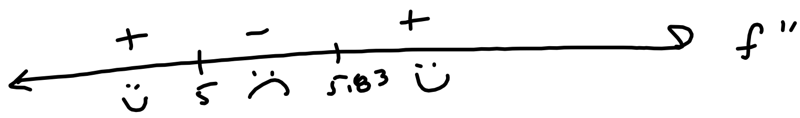
$$f''(x) = \frac{12}{7}(x-5)^{-\frac{2}{7}} + (12x-35)\left(\frac{1}{7}\right)\left(-\frac{2}{7}\right)(x-5)^{-\frac{9}{7}}$$

$$= \frac{12}{7}\left(\frac{1}{(x-5)^{2/7}}\right)\cdot\frac{7(x-5)}{7(x-5)} - \frac{2(12x-35)}{49(x-5)^{9/7}}$$

$$= \frac{12(7)(x-5)-24x+70}{49(x-5)^{9/7}} = \frac{84x-420-24x+70}{49(x-5)^{9/7}}$$

$$= \frac{60x-350}{49(x-5)^{9/7}} = \frac{10(6x-35)}{49(x-5)^{9/7}} = f''(x)$$

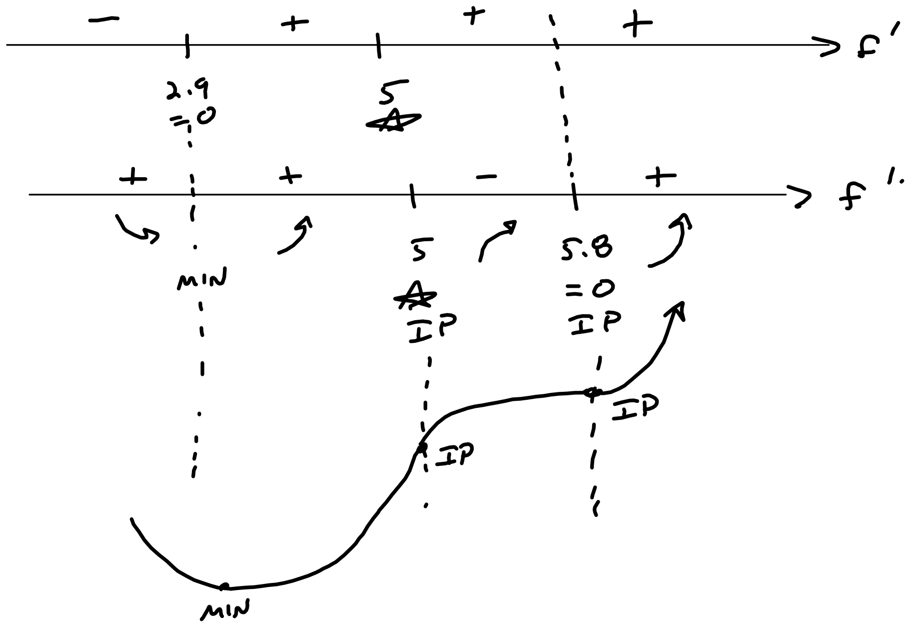
"Critical" : $x = 5, \frac{35}{6} = 5+\frac{5}{6} = 5+.8\overline{33} = 5.8\overline{3}$.



f': $\boxed{2.91\overline{6}, 5 = x}$

2410

MILLS

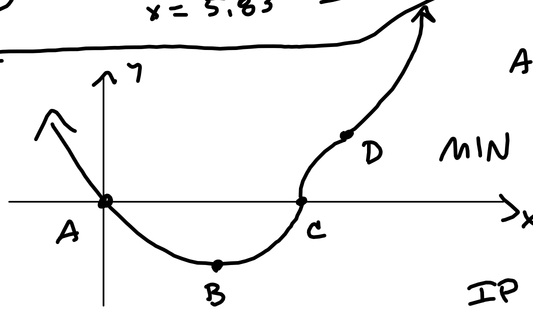


Key points: $x = \frac{35}{12} = 2.91\overline{6}$ MIN

$x = 5$ IP

$x = 5.8\overline{3}$ IP

FINISHED



$A = (0,0)$ x- & y-int.

MIN $B = \left(\frac{35}{12}, f\left(\frac{35}{12}\right)\right) \approx (2.91\overline{6}, -4.927)$

$C = (5, 0)$ x-int, IP

IP $D = \left(\frac{35}{6}, f\left(\frac{35}{6}\right)\right) \approx (5.8\overline{3}, 5.121)$

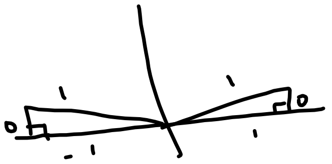
It's really hard to pick up on the 2nd inflection point, visually. We exaggerate it in our graph.

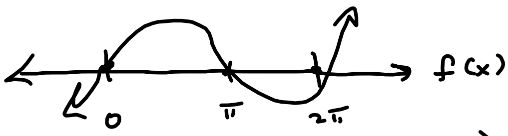
241W MILLS

4) Sol's We sketch a COMPLETE GRAPH of $f(x)=\sin(x)$

$f(x)=\sin(x) \overset{SET}{=} 0 \longrightarrow$

1st dft





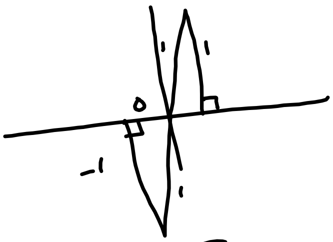
$x=0,\pi,2\pi$

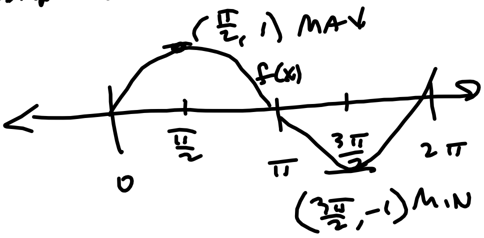
$\sin(\frac{\pi}{2})=1 \longrightarrow +$ on $(0,\pi)$

$\sin(\frac{3\pi}{2})=-1 \longrightarrow -$ on $(\pi,2\pi)$

$f'(x)=\cos(x) \overset{SET}{=} 0 \longrightarrow$

Graph w/ f' info

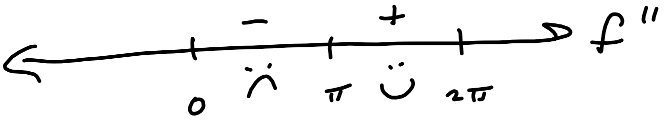




$x=\frac{\pi}{2},\frac{3\pi}{2}$

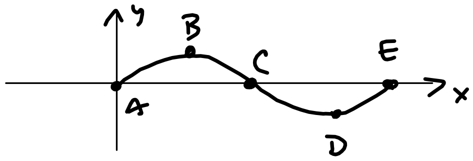
$f''(x)=-\sin(x) \overset{SET}{=} 0$

$\longrightarrow x=0,\pi,2\pi$



Final Graph

$f(x)=\sin(x)$ on $[0,2\pi]$



$A=(0,0)$ x-int, y-int, (IP)

$B=(\frac{\pi}{2},1)$ MAX

$C=(\pi,0)$ x-int, IP

$D=(\frac{\pi}{2},-1)$ MIN

$E=(2\pi,0)$ x-int (IP)

I put "IP" in parentheses at $x = 0$ and $x = 2$pi, because they're endpoints of the interval ovr which we're graphing. Yes, they're inflection points of sin(x), but they're also boundary points of consideration, and it's hard to see that the tangent line is crossing the graph without more graph to look at!