Midterm Solutions, Fall, 2024

1. Evaluate the following limits, if they exist. If one does not exist, explain why.

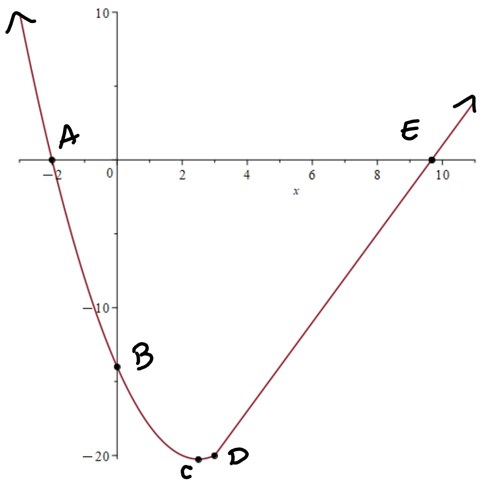
   a. (5 pts) $\lim_{x\to 4^-} \frac{x^2+4x-32}{|x-4|} = \lim_{x\to 4^-} \frac{(x+8)(x-4)}{-(x-4)} = \frac{12}{-1} = \boxed{-12}$

   b. (5 pts) $\lim_{x\to 4^+} \frac{x^2+4x-32}{|x-4|} = \lim_{x\to 4^+} \frac{(x+8)(x-4)}{x-4} = \boxed{12}$

   c. (5 pts) $\lim_{x\to 4} \frac{x^2+4x-32}{|x-4|}$ $\nexists$, b/c $\lim_{x\to 4^-} f = -12 \neq 12 = \lim_{x\to 4^+} f$

2. Consider the piecewise-defined function $f(x) = \begin{cases} x^2-5x-14 & \text{if } x<3 \\ 3x-29 & \text{if } x \geq 3 \end{cases}$.

   a. (5 pts) Sketch the graph of $f(x)$. Label the x- and y-intercepts, the suture point(s), and the vertex of the quadratic piece, if it's in the picture. When I say "Label," I mean an ordered pair, like (0, 5), next to the point.



$A = (-2, 0)$
$B = (0, -14)$
$C = (\frac{5}{2}, -\frac{81}{4}) = (2.5, -20.25)$
$D = (3, -20)$ SUTURE
$E = (\frac{29}{3}, 0) = (9.\bar{6}, 0)$

b. (5 pts) On what interval(s) is $f(x)$ continuous? Explain.

f's 2 pieces are cnt^s polynomials.

$\lim_{x\to 3^-} f(x) = 3^2 - 5(3) - 14 = 9 - 15 - 14 = -20$

$\lim_{x\to 5^+} f(x) = 3(3) - 29 = -20$

$\Rightarrow \boxed{f \text{ is cnt}^s \text{ on } (-\infty, \infty)}$

c. (5 pts) On what interval(s) is $f(x)$ differentiable? Explain.

f is difb^l on its 2 pieces, check the suture point:

From the left: $\lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h} = 2(3) - 5 = 1$

$\lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h} = 2$

$1 \neq 2 \Rightarrow f'(3) \nexists$

$\therefore \boxed{f \text{ is difb}^l \text{ on } (-\infty, 3) \cup (3, \infty)}$

3. (5 pts) Simplify the limit of the difference quotient $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for $f(x)=x^2+2x-10$.

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2+2(x+h)-10-(x^2+2x-10)}{h} =$$

$$= \frac{x^2+2xh+h^2+2x+2h-10-x^2-2x+10}{h}$$

$$= \frac{2xh+h^2+2h}{h} = \frac{2x+h+2}{1} \xrightarrow{h\to 0} \boxed{2x+2 = f'(x)}$$

$(h \neq 0)$

4. The point $P(1,-7)$ lies on the graph of $f(x)=x^2+2x-10$.

a. (5 pts) Write the equation of the tangent line $L_1(x)$ to $f(x)$ at $x=1$.

$f(1)=-7,\ f'(x)=2x+2 \Rightarrow f'(1) = 2+2 = 4 = f'(1).$

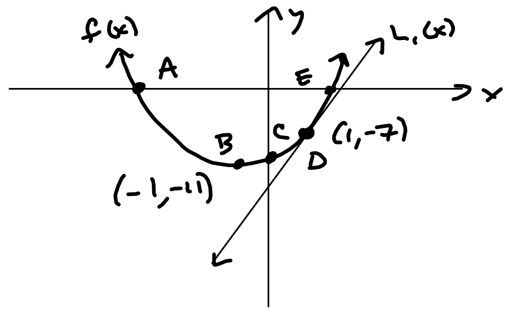
$L_1(x) = f'(1)(x-1)+f(1)$

$\boxed{L_1(x) = 4(x-1)-7}$ STOP!

$= 4x-4-7 = 4x-11$, for students who don't listen.

b. (5 pts) Sketch a graph of $f(x)$ and the tangent line to $f(x)$ at the point P.

$f(x) = x^2+2x-10 = x^2+2x+1-1-10 = (x+1)^2-11$



$A = (-1-\sqrt{11}, 0) \approx$ (-4.31662, 0)

$B = (-1,-11)$

$C = (0,-10)$

$D = (1,-7)$

$E = (-1+\sqrt{11}, 0) \approx$ (2.31662, 0)

$(x-1)^2-11 = 0$

$(x-1)^2 = 11$

$x = 1 \pm \sqrt{11}$

5. (5 pts) Prove that $\lim_{x \to -2}(5x+7) = -3$, using the $\varepsilon - \delta$ definition of limit.

Proof

Define $f(x) = 3x-7$ and $L = -1$. Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{5}$

Then $0 < |x-2| < \delta \Rightarrow |f(x) - L| = |5x+7-(-3)| = |5x+10|$

$= 5|x+2| < 5\delta = \varepsilon$ ■

6. (5 pts) Prove that the equation $f(x) = x^4 - 4x^3 + 6x^2 + 28x - 91$ has a root in the interval $(0,5)$, but *do not solve*!

$f(0) = -91$

$f(5) = 324$

$$\begin{array}{r|rrrrr} 5 & 1 & -4 & 6 & 28 & -91 \\ & & 5 & 5 & 55 & 415 \\ \hline & 1 & 1 & 11 & 83 & 324 \end{array}$$

f is a polynomial, hence cont's $[0,5]$

By IVT, the fact that $f(0) = -91 < 0$ & $f(5) = 324 > 0$, implies.

$\exists\ c \in (0,5) \ni f(c) = 0$.

7. Differentiate the following with respect to the indicated independent variable. **Do not simplify!**

a. (5 pts) $f(x)=\sqrt[3]{x^5}-27x^{\frac{7}{4}}+-\frac{11}{x^6}$; x.

$f(x)= x^{\frac{5}{3}} - 27x^{\frac{7}{4}} - 11x^{-6}$ ⟶

$\boxed{f'(x)= \frac{5}{3}x^{\frac{2}{3}} - \frac{27(7)}{4}x^{\frac{3}{4}} + 66x^{-7}}$

b. (5 pts) $g(x)=\cos(5x)\sec(3x)$; x.

⟶ $\boxed{g'(x)= -5\sin(5x)\sec(3x) + 3\cos(5x)\sec(3x)\tan(3x)}$

c. (5 pts) $h(\beta)=\frac{12\beta^3+2\beta}{\tan(\beta)}$; β. ⟶

$$\boxed{h'(\beta)= \frac{(36\beta^2+2)(\tan(\beta)) - (12\beta^3+2\beta)\sec^2(\beta)}{\tan^2(\beta)}}$$

d. (5 pts) $r(w)=(w^2+11w+5)^4(2w+6)^3$; w. ⟶

$$\boxed{r'(w)= 4(w^2+11w+5)^3(2w+6)^3 + (w^2+11w+5)(3)(2w+6)^2(2)}$$

10. Consider the relation $x^2 - 3xy + 4y^2 = \cos(y)$.

a. (5 pts) Use implicit differentiation to find $y' = \dfrac{dy}{dx}$

$$2x - 3y - 3xy' + 8yy' = -\sin(y)\,y'$$

$$(-3x + 8y + \sin(y))\,y' = -2x + 3y$$

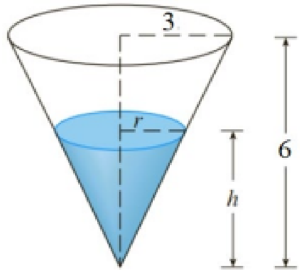
$$\boxed{y' = \frac{-2x+3y}{-3x+8y+\sin(y)}}$$

b. (5 pts) Find an equation of the tangent line to the curve at the point $(1,0)$. $= (x_1, y_1)$

$$y'\Big|_{(x,y)=(1,0)} = \frac{-2(1)+3(0)}{-3(1)+8(0)-\sin(0)} = \frac{-2}{-3} = \frac{2}{3} = m$$

$$\longrightarrow \boxed{y = \frac{2}{3}(x-1) + 0}$$

9. (10 pts) A water tank has the shape of an inverted circular cone with base radius 3 m and height 4 m. If water is being pumped out of the tank at a rate of 3 $\dfrac{\text{m}^3}{\text{min}}$, find the rate at which the water level is dropping when the water is 4 m deep. Hints: The volume of a circular cone is $V = \dfrac{1}{3}\pi r^2 h$. It is possible (and essential) to write V as a function of r for this situation.



The hint sucks.

Better version: **("Worse" version on the following page)**

Let h = height of the water in m.
r = radius of the water surface in m.
V = volume of water in the cone, in m^3.

We want $\left.\dfrac{dh}{dt}\right|_{h=4}$. Thus we want to replace r by an expression in h:

$$\frac{r}{h} = \frac{3}{6} = \frac{1}{2} \Rightarrow 2r = h \Rightarrow r = \frac{h}{2} \Rightarrow$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3 \Rightarrow$$

$$\left.\frac{dV}{dt}\right|_{h=4} = \left.\frac{\pi}{12}(3h^2)\frac{dh}{dt}\right|_{h=4} = \left.\frac{\pi}{4}h^2\frac{dh}{dt}\right|_{h=4} = \frac{\pi}{4}(4)^2\frac{dh}{dt} = -3 \Rightarrow$$

$$4\pi\left.\frac{dh}{dt}\right|_{h=4} = -3 \Rightarrow \left.\frac{dh}{dt}\right|_{h=4} = \boxed{\frac{-3}{4\pi}\ \frac{m}{min} = \left.\frac{dh}{dt}\right|_{h=4}}$$

≈ -0.238732414638

Following the terrible hint, we express the problem in terms of the radius *r*, which necessitates finding *dh*/*dt* after finding *dr*/*dt*.

By similar triangles, $\frac{r}{h} = \frac{3}{6} = \frac{1}{2} \Rightarrow$

$2r = h \Rightarrow$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (2r) = \frac{2}{3}\pi r^3$$

$$\Rightarrow \frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt} = \frac{3m^3}{min}$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{h=4} = 2\pi r^2 \frac{dr}{dt} = 3. \quad \text{Find } r\Big|_{h=4} = \frac{h}{2}\Big|_{h=4} = 2$$

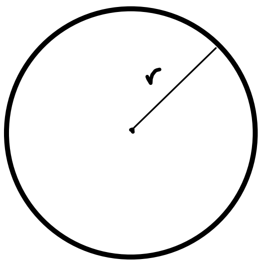
$$\Rightarrow 2\pi (2)^2 \left(\frac{dr}{dt}\right) = 3$$

$$\Rightarrow \frac{dr}{dt} = \frac{3}{8\pi}\frac{m}{min}, \text{ but we want } \frac{dh}{dt}!$$

$h = 2r \Rightarrow$

$$\frac{dh}{dt} = 2\frac{dr}{dt} = 2\left(\frac{3}{8\pi}\right) = \boxed{\frac{3}{4\pi}\frac{m}{min} = \left.\frac{dh}{dt}\right|_{h=4}}$$

10. (10 pts) If the minute hand of a clock has length r (in centimeters), find the rate at which it sweeps out area as a function of r.



Let A = area swept by the minute hand as a function of t = time, in hours.

We want $\frac{dA}{dt}$ as a function of r

$$\left(\frac{1 \text{ rev}}{\text{hr}}\right)\left(\frac{2\pi \text{ radians}}{1 \text{ rev}}\right) = \frac{d\theta}{dt} = 2\pi \frac{\text{radians}}{\text{hr}} \text{ or simply } \frac{2\pi}{\text{hr}}$$

Area $= \frac{1}{2}r^2\theta$, r is constant!

$$\frac{dA}{dt} = r\theta\frac{dr}{dt} + \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2(2\pi) = \boxed{\pi r^2 \frac{\text{cm}^2}{\text{hr}}}$$

11. Let $R(x) = \dfrac{(x-2)(x-5)}{x-8} = \dfrac{x^2-7x+10}{x-8}$

a. (5 pts) What is the domain of R? Give any vertical asymptotes for R.

$\boxed{\mathcal{D}(R) = \mathbb{R} \setminus \{8\}}$ $\boxed{x = 8 \text{ is V.A.}}$

b. (5 pts) Find the slant asymptote for R.

$$\begin{array}{r|rrr} 8 & 1 & -7 & 10 \\ & & 8 & 8 \\ \hline & 1 & 1 & 18 \end{array}$$

$\boxed{y = x+1}$ is slant asymptote.

c. (5 pts) Find $R'(x)$. Find its zeros and any points where it blows up.

$$R'(x) = \frac{(2x-7)(x-8)-(x^2-7x+10)(1)}{(x-8)^2} = \frac{2x^2-16x-7x+56-x^2+7x-10}{(x-8)^2}$$

$$R'(x) = \frac{x^2-16x+46}{(x-8)^2} \overset{\text{SET}}{=} 0 \Rightarrow x^2-16x+8^2-64+46 = (x-8)^2-18 = 0$$

$\Rightarrow x = 8 \pm \sqrt{18} = 8 \pm 3\sqrt{2}$

$\Rightarrow$ $R'(x) = 0$ @ $x = 8 \pm 3\sqrt{2}$
$R'(x) \nexists$ @ $x = 8$

wasn't asked

$R(x) = 0$ @ $x = 2, 5$
$R(x) \nexists$ @ $x = 8$

d. (5 pts) Create a sign pattern for $R'(x)$.

$8 + 3\sqrt{2} \approx 12.24$

$8 - 3\sqrt{2} \approx 3.76$

R': $+$ | 3.76 ($=0$) | $-$ | 8 ($\nexists$) | $-$ | 12.24 ($=0$) | $+$

e. (5 pts) Find $R''(x)$.

$$R'(x) = \frac{x^2-16x+46}{(x-8)^2}$$

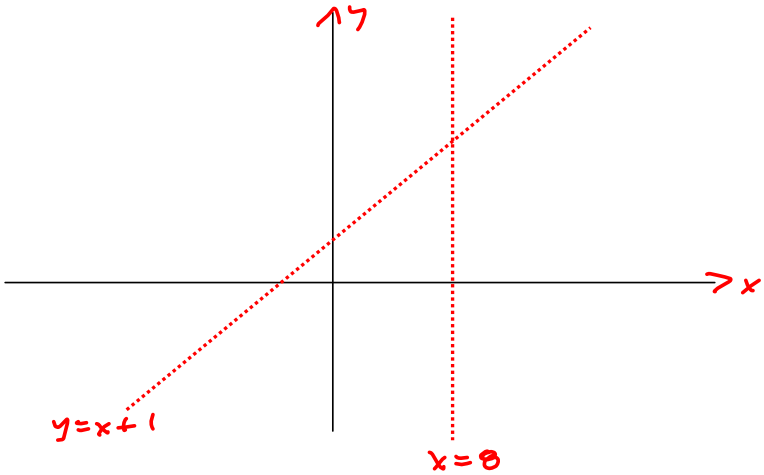
$$R''(x) = \frac{(2x-16)(x-8)^2-(x^2-16x+46)(2(x-8))}{(x-8)^4} = \frac{(2x-16)(x-8)-2(x^2-16x+46)}{(x-8)^3}$$

$$= \frac{2x^2-16x-16x+128-2x^2+32x-92}{(x-8)^3} = \boxed{\frac{36}{(x-8)^3} = R''(x)}$$

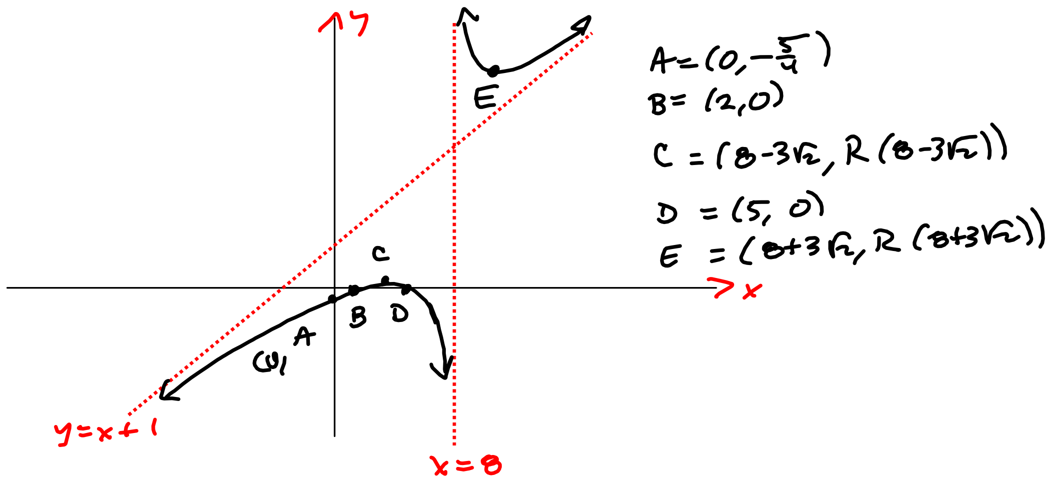
g. (5 pts) Create a sign pattern for $R''(x)$.

$R''(x)$: $-$ | 8 | $+$

h. (5 pts) Sketch a graph of the Asymptotes of R.



i. **(Bonus 10 pts)** Sketch a complete graph of R. If you encounter a max/min or inflection point where the x-value is something like $x = 7 - 5\sqrt{3}$, it will suffice to write $B = \left(7 - 5\sqrt{3}, R\left(7 - 5\sqrt{3}\right)\right)$ in a list of points, A, B, C, etc., off to the side, with the points on the graph labeled A, B, C, etc., as long as the points are generally in the right location. I don't want you to have to evaluate messy functions with messy inputs, here.



Bonus

1. (5 pts) Prove that $\lim_{x\to 2}\left(3x^2-2x-5\right)=3$, using the $\varepsilon-\delta$ definition of limit.

$3(4)-4-5=3$ ✓

Let $f(x)=3x^2-2x-5$ & $L=5$

Let $\varepsilon>0$ be given. Assume $\delta\le 1$. Then

$0<|x-2|<\delta \longrightarrow |f(x)-L| = |3x^2-2x-5-3|$ $\qquad 3(8)=\underline{3\cdot 2}\cdot 2\cdot 2$

$=|3x^2-2x-8| = |3x^2-6x+4x-8| = |3x(x-2)+4\cdot(x-2)|$

$=|3x+4||x-2| < \underbrace{|3x+4|}\delta$

Need a bound on this in $(2-\delta, 2+\delta)$

$|x-2|<1 \longrightarrow$

$-1<x-2<1 \longrightarrow$

$1<x<3 \longrightarrow$

$3<3x<9$

$7<3x+4<13$

$\longrightarrow |3x+4|<13$. Define $\delta=\min\left\{1,\frac{\varepsilon}{13}\right\}$

Proof

Let $\varepsilon>0$ be given. Define $\delta=\min\left\{1,\frac{\varepsilon}{13}\right\}$. Then $0<|x-2|<\delta$

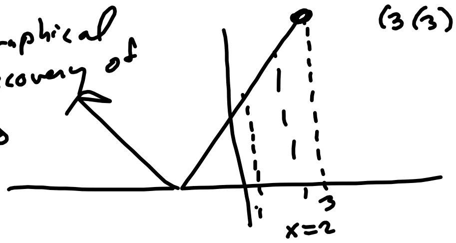
implies $|f(x)-L| = |3x^2-2x-8| = |3x^2-6x+4x-8| = |3x(x-2)+4\cdot(x-2)|$

$=|3x+4||x-2| < |3x+4|\,\delta < 13\delta \le 13\cdot\frac{\varepsilon}{13}=\varepsilon$ ∎

$(3(3)+4)=13$

$\rightarrow$ what we need!

Graphical discovery of δ



2. (5 pts) Compute the derivative of $f(x) = x^{\frac{4}{3}}$ by the limit definition of the derivative. Hint: Use this formulation: $\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}$ and be mindful of special products.

Difference of two cubes factoring model.

$$x^{\frac{4}{3}} - c^{\frac{4}{3}} = (x^4)^{\frac{1}{3}} - (c^4)^{\frac{1}{3}}$$

$$(x-y)(x^2+xy+y^2) = x^3-y^3$$

$$= \frac{(x^{\frac{4}{3}}-c^{\frac{4}{3}})(x^{\frac{8}{3}}+(cx)^{\frac{4}{3}}+c^{\frac{8}{3}})}{(x-c)(x^{\frac{8}{3}}+(cx)^{\frac{4}{3}}+c^{\frac{8}{3}})} = \frac{x^4-c^4}{(x-c)(\sim\sim)}$$

$$= \frac{(x^2-c^2)(x^2+c^2)}{\sim\sim} = \frac{(x-c)(x+c)(x^2+c^2)}{(x-c)(x^{8/3}+(cx)^{4/3}+c^{8/3})} = \frac{(x+c)(x^2+c^2)}{x^{8/3}+c^{4/3}x^{4/3}+c^{8/3}}$$

$(x \neq c)$

$$\xrightarrow{x\to c} \frac{2c(2c^2)}{c^{8/3}+c^{8/3}+c^{8/3}} = \frac{4c^3}{3c^{8/3}} = \frac{4c^{\frac{1}{3}}}{3}, \text{ i.e.,}$$

$$\frac{d}{dx}\left[x^{\frac{4}{3}}\right] = \frac{4}{3}x^{\frac{1}{3}} \ !$$

3. (5 pts) Find the x-intercept of the tangent line $L_{x_1}(x)$ to a differentiable function $f(x)$ at x_1

$L_{x_1}(x) = f'(x_1)(x-x_1) + f(x_1) \overset{\text{SET}}{=} 0 \longrightarrow$

$f'(x_1)(x-x_1) = -f(x_1)$

$x - x_1 = -\dfrac{f(x_1)}{f'(x_1)} \longrightarrow$

$\boxed{x = x_1 - \dfrac{f(x_1)}{f'(x_1)}} = x_2$

4. (5 pts) If $x_1 = \dfrac{3\pi}{4}$ and $f(x) = 2\sin(3x) + 5\sin(2x)$, what does Newton's Method say x_2 is?

$f(x_1) = f\left(\frac{3\pi}{4}\right) = 2\sin\left(3\left(\frac{3\pi}{4}\right)\right) + 5\sin\left(2\left(\frac{3\pi}{4}\right)\right)$

$= 2\sin\left(\frac{9\pi}{4}\right) + 5\sin\left(\frac{3\pi}{2}\right)$

$= 2\left(\frac{\sqrt{2}}{2}\right) + 5(-1) = \boxed{\sqrt{2} - 5 = f(x_1)}$

$f'(x) = 6\cos(3x) + 10\cos(2x) \longrightarrow$

$f'(x_1) = f'\left(\frac{3\pi}{4}\right) = 6\cos\left(\frac{9\pi}{4}\right) + 10\cos\left(\frac{3\pi}{2}\right)$

$= 6\left(\frac{\sqrt{2}}{2}\right) + 0 = \boxed{3\sqrt{2} = f'(x_1).}$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = \boxed{\dfrac{3\pi}{4} - \dfrac{\sqrt{2}-5}{3\sqrt{2}} = x_2} \approx 3.201372459$

3. (5 pts) See if you can *squeeze* out a *convincing* argument to support the statement

$$f(x)=\begin{cases} x^2\sin\left(\dfrac{\pi}{x}\right) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases} \text{ is continuous on } (-\infty,\infty).$$

x^2 is cnt^s everywhere.

$\sin\left(\frac{\pi}{x}\right)$ is composition of a function that's cnt^s everywhere, with a function that's cnt^s everywhere, except $x=0$.

So $f(x)$ is cnt^s on $(-\infty,0)\cup(0,\infty)$.

Continuity at $x=0$ is trickier. For this, it will suffice to show that $\lim\limits_{x\to 0} f(x) = f(0) = 0$.

To this end, observe that $-1 \le \sin\left(\frac{\pi}{x}\right) \le 1 \quad \forall\, x\neq 0$.

since $x^2>0 \;\forall\, x\neq 0$,

$$-x^2 \le x^2\sin\left(\frac{\pi}{x}\right) \le x^2 \quad \forall\, x\neq 0$$

$$\underset{\substack{\downarrow \\ 0}}{\underset{x\to 0}{-x^2}} \qquad \underset{x\to 0}{x^2\sin\left(\tfrac{\pi}{x}\right)} \qquad \underset{\substack{\downarrow \\ 0}}{\underset{x\to 0}{x^2}}$$

By the Squeeze Theorem,

$\lim\limits_{x\to 0} f(x) = 0$. This means the limit agrees with the function value @ $x=0$, which is the definition of continuity at $x=0$. $\therefore$ $f(x)$ is cnt^s @ $x=0$ & so $f(x)$ is cnt^s $\forall\, x\in(-\infty,\infty)$ ∎

6. (5 pts) Let $f(x)=x^3-6x^2+5x-4$ Find $c\in(0,4)$ such that $f'(c)$ equals the average slope of f over the interval $[0,4]$. How did you know such a c existed before you started?

f is a polynomial & $\therefore$ cnt's and difbl on $(-\infty,\infty)$.

$\Rightarrow$ f is cnt's on $[0,4]$ & difbl on $(0,4)$ $\Rightarrow$ hypotheses of MVT are satisfied $\Rightarrow \exists\, c\in(0,4) \ni f'(c)=\dfrac{f(4)-f(0)}{4-0}$

$$\begin{array}{r|rrrr} 4 & 1 & -6 & 5 & -4 \\ & & 4 & -8 & -12 \\ \hline & 1 & -2 & -3 & -16=f(4) \end{array}$$

$$\frac{f(4)-f(0)}{4-0}=\frac{-16-(-4)}{4}=\frac{-12}{4}=\boxed{-3=m_{AVG}}$$

$f'(x)=3x^2-12x+5 \overset{SET}{=} m_{avg}=-3 \Rightarrow$

$3x^2-12x+8=0$

$\Rightarrow 3(x^2-4x+2^2)-3(4)+8$

$=3(x-2)^2-4 \overset{SET}{=} 0 \Rightarrow$

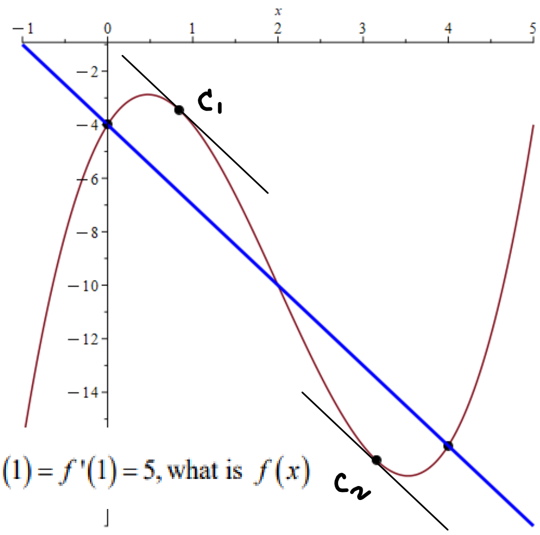
$3(x-2)^2=4 \Rightarrow$

$(x-2)^2=\frac{4}{3} \Rightarrow$

$x=2\pm\frac{2}{\sqrt{3}}=2\pm\frac{2\sqrt{3}}{3}$

$\Rightarrow c=2\pm\frac{2\sqrt{3}}{3}$

There are 2 of them!



7. (5 pts) Suppose $f''(x)=20x^3-12x^2+6x-4$. Given $f(1)=f'(1)=5$, what is $f(x)$

$f''(x)=20x^3-12x^2+6x-4 \Rightarrow$

$f'(x)=5x^4-4x^3+3x^2-4x+C \Rightarrow f'(1)=5-4+3-4+C=5 \Rightarrow \boxed{C=5}$

$\boxed{f(x)=x^5-x^4+x^3-2x^2+Cx+D} \Rightarrow$

$f(1)=1-1+1-2+5+D=5$

$\Rightarrow 4+D=5$

$\Rightarrow \boxed{D=1}$