Review on exponentials and logarithms

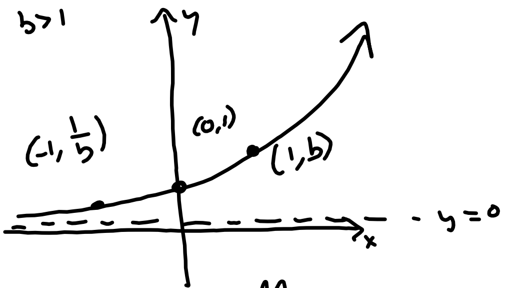
Convince you what *e* is and why.

$f(x) = b^x, \quad b > 0$

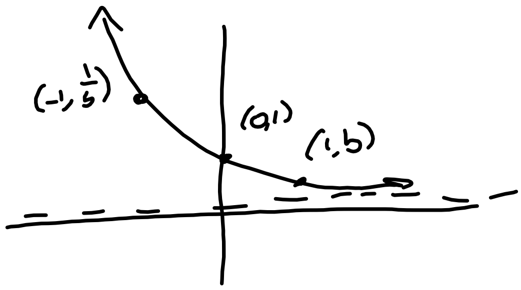
$b = \frac{1}{3}$

Exponential Decay.

$b > 1$



$0 < b < 1$



Exponentially

$f(x) = 3^x \Rightarrow$

$f(-x) = 3^{-x} = \left(\frac{1}{3}\right)^x$

$3^{-x} = 3^{-1 \cdot x} = (3^{-1})^x = \left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x}$

Properties:

$D(b^x) = \mathbb{R}$

$R(b^x) = (0, \infty)$

H.A. $y = 0$

$b^{xy} = (b^x)^y = (b^y)^x$

$b^a b^c = b^{a+c}$

$\frac{b^a}{b^c} = b^{a-c}$

$\log(b^y) = y\log(b)$

$\log(xy) = \log(x) + \log(y)$

$\log(x^3) = \log(x \cdot x \cdot x)$
$= \log(x) + \log(x) + \log(x)$
$= 3\log(x)$

"e" = the natural base, named for Euler.

e is the base such that

$\frac{d}{dx}[e^x] = e^x$

$\frac{e^{x+h}-e^x}{h} = \frac{e^x e^h - e^x}{h} = \frac{e^x(e^h-1)}{h} = \left(\frac{e^h-1}{h}\right)e^x \xrightarrow{h\to 0} e^x$

"e" is the number such that

$\lim_{h\to 0} \frac{e^h-1}{h} = 1$

e ≈ 2.718268, by Desmos Guess-and-Check.

$f(h) = \frac{(2.718268^h - 1)}{h}$

$f(0.00001)$

$= 0.999999912743$

Change of Base

The inverse function for $f(x) = b^x$ is $\log_b(x)$.

$y = \log_b(x)$ means

$x = b^y$

$\log_3 81 = 4$, b/c $81 = 3^4$

$\log_3(3^4) = 4$

$100 = 10^2$

$10000 = 10^5$

$100 \cdot 10000 = 10^2 \cdot 10^5 = 10^{2+5} = 10^7$

$\log_{10}(10^2 \cdot 10^5) = 2+5 = 7$

$y = \log_b(x)$

$b^y = x$

$\log_c(b^y) = \log_c(x)$

$y \log_c(b) = \log_c(x)$

$y = \dfrac{\log_c(x)}{\log_c(b)} = \log_b(x)$

$\ln(x) = \log_e(x)$

$\frac{d}{dx}\left[3^x\right]$

$3^x = ?\, e^x$

$x = \ln(e^x) = \log_3(3^x)$

$3^x = 3^{\ln(e^x)} = 3^{x \cdot \ln(e)}$

$y = 3^x =$

$\ln(y) = \ln(3^x) = x\ln(3)$

$e^{\ln(y)} = y = e^{x\ln(3)} = e^{\ln(3)x} = \left(e^{\ln(3)}\right)^x = 3^x$

$\frac{d}{dx}\left[3^x\right] = \frac{d}{dx}\left[e^{\ln(3)x}\right]$

$= \frac{d}{dx}\left[e^{u(x)}\right]$, where $u(x) = \ln(3)x$

$= \frac{d}{du}\left(e^{u(x)}\right) \cdot u'(x) = e^{\ln(3)x} \cdot \ln(3) = 3^x \cdot \ln(3)$

$\boxed{\frac{d}{dx}\left[b^x\right] = \ln(b) \cdot b^x}$

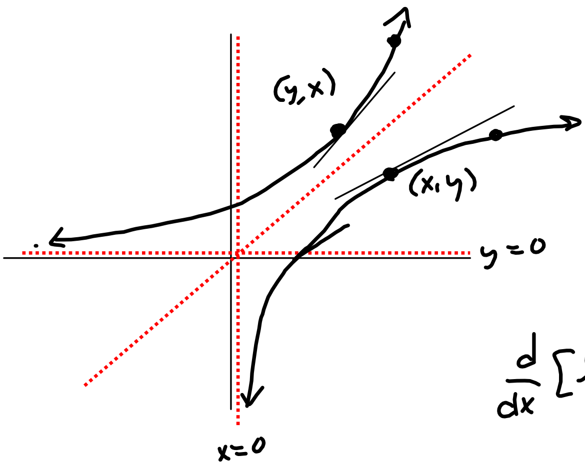
$\frac{d}{dx}\left[\sin(x^2)\right] = \frac{d}{dx^2}\left(\sin(x^2)\right) \cdot \frac{d}{dx} x^2$

$= \cos(x^2) \cdot 2x$

Derivative of the inverse

$$y = f^{-1}(x)$$

$$\longrightarrow f(y) = x$$



The tangent lines are reciprocals.

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$$

$$\frac{d}{dx}\left[\ln(x)\right] =$$

$f^{-1}(x) = \ln(x)$

$f(x) = e^x$

$f'(x) = e^x$

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{e^{\ln(x)}}$$

$$= \frac{1}{x} \quad ?!!$$

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$$

$$\frac{d}{dx}\left[\ln(g(x))\right] = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$$

$$\frac{d}{dx}\left[e^x\right] = e^x \qquad \frac{d}{dx}\left[e^{u(x)}\right] = e^{u(x)} \cdot u'(x)$$

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x} \qquad \frac{d}{dx}\left[\ln(u(x))\right] = \frac{u'(x)}{u(x)}$$

$$\int \frac{dx}{x} = \ln(x) + C \quad \text{for } x > 0$$

$$\int \frac{dx}{x} = \ln(-x) + C \quad \text{for } x < 0$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \tan(x)\,dx = \int \frac{\sin(x)\,dx}{\cos(x)}$$

Let $u = \cos(x)$

$du = -\sin(x)\,dx$ $\quad -\int \frac{du}{u} = -\ln|u| + C$

$$= -\ln|\cos(x)| + C$$

$$= \ln\left(|\cos(x)|^{-1}\right| + C$$

$$= \ln\left|\frac{1}{\cos(x)}\right| + C = \boxed{\ln|\sec(x)| + C = \int \tan(x)\,dx}$$

$$\int \sec(x)\,dx = \int \sec(x)\left(\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\right)dx$$

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx = \int \frac{du}{u}$$

where $u = \sec(x) + \tan(x)$ and
$du = (\sec^2(x) + \sec(x)\tan dx)\,dx$

$$= \ln|u| + C = \ln|\sec(x) + \tan(x)| + C$$

Exercise: Do same for $\int \cot(x)\,dx$ & $\int \csc(x)\,dx$

$f(x) = x^5 - x^7 + 5x^6 = 5 \qquad f(1) = 5$

$f^{-1}(5) = 1$

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(1)} = \frac{1}{28}$$

$f'(x) = 5x^4 - 7x^6 + 30x^5$

$f'(1) = 5 - 7 + 30 = 28$

Finding $f^{-1}(x)$ in general is often un-possible.