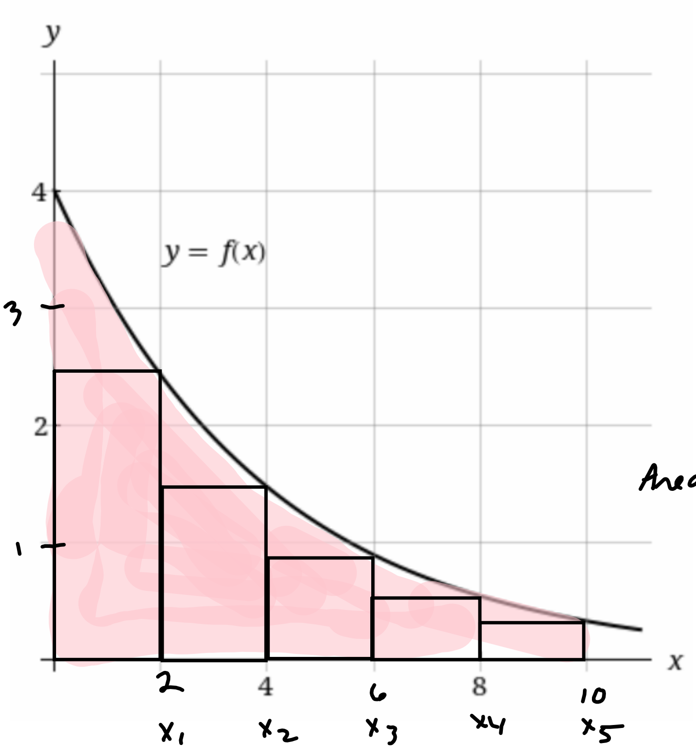


Use $n = 5$ rectangles to estimate the area under the curve.

Use a lower estimate. Since f is decreasing, the right endpoints are the low points, yielding the lower estimate.

$$\text{Area} \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$$

$\Delta x = \frac{10-0}{5} = 2 =$ width of rectangles

$$\text{Area} \approx 2.4\cdot 2 + 1.5\cdot 2 + .8(2) + .5(2) + .3(2)$$

$$= (2.4 + 1.5 + .8 + .5 + .3)(2)$$

$$= (f(x_1) + f(x_2) + \cdots + f(x_5))\Delta x$$

$$= \Delta x\,(f(x_1) + f(x_2) + \cdots + f(x_5))$$

$$= \Delta x \sum_{k=1}^{5} f(x_k)$$

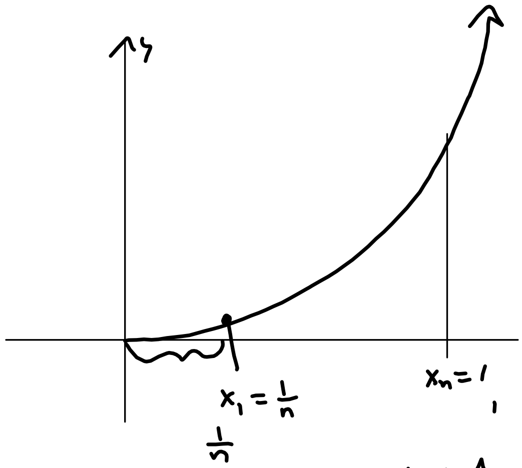
= Right-endpoint Riemann Sum for $f(x)$ on $[a,b] = [0,10]$ using width $= \Delta x = \frac{b-a}{n} = \frac{10-0}{5} = \frac{10}{5} = 2$

Notice as $n \to$ big

$\Delta x = \frac{b-a}{n} \to$ small

we use rectangles of the same width.

$f(x) = x^2$. Find the area under $f(x)$ from $x=0$ to $x=1$



Right endpoints
Let n be unfixed, for now.

$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

$\Delta x\left(f(x_1) + f(x_2) + \dots + f(x_n)\right)$

$\frac{1}{n}\sum_{k=1}^{n} f(x_n) = \frac{1}{n}\sum_{k=1}^{n} f\left(k\cdot\frac{1}{n}\right)$

$= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2$

$= \frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2}$

$= \frac{1}{n^3}\sum_{k=1}^{n} k^2$

$x_2 = x_1 + \Delta x = \frac{1}{n} + \frac{1}{n} = 2\cdot\frac{1}{n}$

$x_3 = 3\cdot\frac{1}{n}$

$\vdots$

$x_n = 1 = n\cdot\frac{1}{n}$

$\sum_{k=1}^{n} 1 = \underbrace{1+1+1+1+\dots+1}_{n} = n$

$\sum_{k=1}^{n} k = 1+2+3+\dots+n = \frac{n(n+1)}{2}$

$\sum_{k=1}^{100} k = 5050$

$1+2+3+\dots+97+98+99+100 = 50(101)$

$= \frac{100(101)}{2}$

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + \text{lower degree}}{6} = \frac{n^3}{3} + \frac{\text{lower stuff}}{6}$

<u>Proof by Induction</u>

Prove it holds for $n=1$

$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 1$ so it works for $n=1$.

Assume it works for some $n \geq 1$, $\sum_{k=1}^{n} k = \frac{n(n+1)(2n+1)}{6}$ we show $\sum_{k=1}^{n+1} k^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$

$$\sum_{k=1}^{n+1} k^2 = \underbrace{1^2+2^2+3^2+\dots+(n-1)^2+n^2}_{\sum_{k=1}^{n} k^2} + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n(2n^2+3n+1)}{6} + \frac{(n^2+2n+1)6}{6}$$

$$= \frac{2n^3+3n^2+n}{6} + \frac{6n^2+12n+6}{6} = \frac{2n^3+9n^2+13n+6}{6}$$

$$= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \Longrightarrow \text{it works for } n+1!$$

$n+1 \overline{)\, 2n^3+9n^2+13n+6}$

$$\begin{array}{r|rrrr} -1 & 2 & 9 & 13 & 6 \\ & & -2 & -7 & -6 \\ \hline & 2 & 7 & 6 & 0 \end{array}$$

$$(n+1)(2n^2+7n+6)$$
$$= (n+1)(2n+3)(n+2)$$
$$= (n+1)(n+2)(2n+3)$$
$$= (n+1)((n+1)+1)(2n+2+1)$$
$$= (n+1)((n+1)+1)(2(n+1)+1)$$

$$(n(n+1)(2n+1))$$
$$= n(2n^2+3n+1) = 2n^3+3n^2+n$$

Area under x^2 from 0 to 1 is

$$A \approx \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{1}{n^3}\left[\frac{2n^3 + \text{lower degree}}{6}\right]$$

$$= \frac{1}{n^3}\left[\frac{n^3}{3} + \frac{\text{smaller degree}}{6}\right] = \left[\frac{1}{3} + \frac{\#n^2}{6n^3} + \frac{\#n}{6n^3} + \frac{\#}{6n^3}\right]$$

$\to 0$ $\to 0$ $\to 0$

$$\xrightarrow{n\to\infty} \boxed{\frac{1}{3} = \text{AREA EXACTLY.}}$$

upper limit

FTC II

Differential

$$\int_0^1 x^2\,dx = \text{Area under } x^2 = \left.\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

Integrand

Lower Limit

$$\left(\sum x_k^2\right)\Delta x = \left(\sum f(x_k)\right)\Delta x$$

$$= \sum\left(f(x_k)\Delta x\right)$$