Limit Laws

Assume $\lim\limits_{x\to c} f(x) = L$, $\lim\limits_{x\to c} g(x) = M$

and $a, b \in \mathbb{R}$.

Then $\lim (af + bg) = a \lim f + b \lim g = aL + bM$

$\lim (fg) = (\lim f)(\lim g) = LM$

$\lim \left(\frac{f}{g}\right) = \frac{\lim f}{\lim g} = \frac{L}{M}$, provided that $\lim g \neq 0$

Assume $n \in \mathbb{Z}$

Then $\lim (f^n) = L^n$

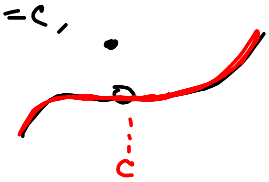
(Exception: $\lim f = 0$ AND $n < 0$)

and $\lim \sqrt[n]{f} = \sqrt[n]{\lim f} = \sqrt[n]{L}$, provided $\sqrt[n]{L}$ $\exists$.

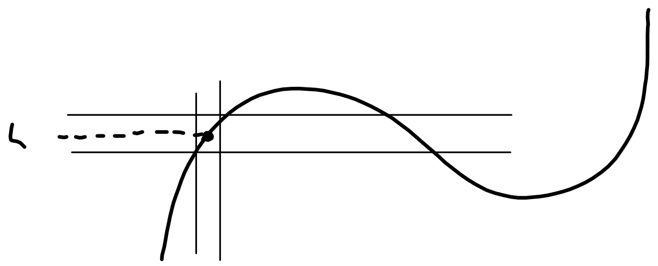
$f(x) \leq g(x)$

If $f(x) = g(x)$ except possibly at $x = c$,

then $\lim\limits_{x\to c} f \overset{\leq}{=} \lim\limits_{x\to c} g$



$\lim\limits_{x\to 1} \frac{x^2-1}{x-1} = \lim\limits_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim\limits_{x\to 1} (x+1) = 2$



Claim: $\lim_{x \to 3} (7x-5) = 16$

Scratch: Want: $|f(x) - L| = |7x-5-16|$

$= |7x-21| = 7|x-3| < \varepsilon$

$|x-3| < \frac{\varepsilon}{7}$

<u>Proof</u> Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{7}$.

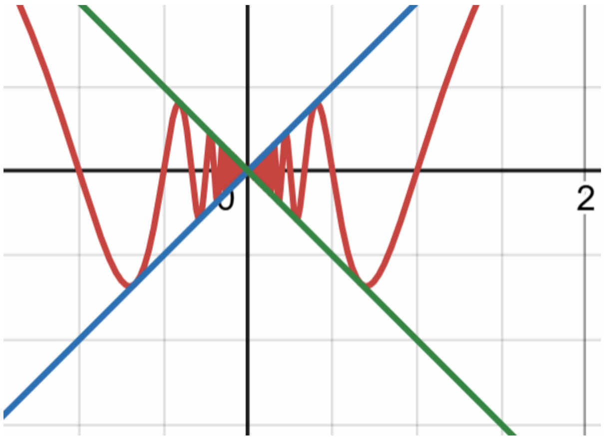
Then $0 < |x-3| < \delta$ implies

$|f(x) - L| = |7x-5-16| = |7x-21| = 7|x-3| < 7\delta$

$= 7 \cdot \frac{\varepsilon}{7} = \varepsilon$. $\blacksquare$

5. (5 pts) Prove $f(x) = x\sin\left(\frac{\pi}{x}\right)$ has a *removable discontinuity* at $x = 0$, namely,

$$g(x) = \begin{cases} x\sin\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$ is a continuous function that agrees with f everywhere except at one point, but by filling in that one point, we remove the hole in f. This is not an $\varepsilon - \delta$ proof. Instead, you may use facts about continuity of polynomials, rational functions, and trigonometric functions for most of it. The trick part will require the application of **The Squeeze Theorm**.



$$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1 \quad (x \neq 0)$$

$$-x \le x\sin\left(\frac{\pi}{x}\right) \le x$$

$$\underset{x\to 0}{\downarrow} \qquad\qquad \underset{x\to 0}{\downarrow}$$

$$0 \qquad\qquad\qquad 0$$

$$\Rightarrow \lim_{x\to 0} x\sin\left(\frac{\pi}{x}\right) = 0$$

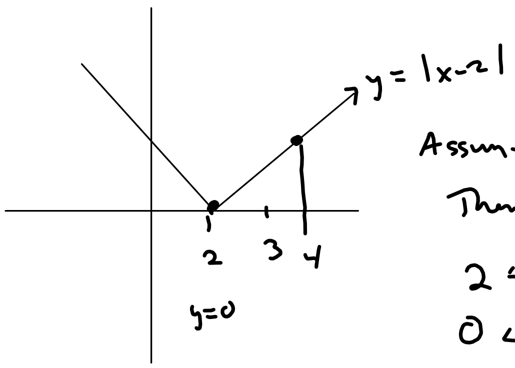
by the squeeze theorem.

Claim $\lim_{x \to 3} (x^2 - 5x + 4) = -2$

Scratch: Want $|f(x) - L| < \varepsilon$

$$|x^2-5x+4-(-2)| = |x^2-5x+6| = \underbrace{|x-2|}_{\text{Need a bound}} \underbrace{|x-3|}_{<\delta}$$

on this in a neighborhood of $x=3$



Assume $\delta \leq 1$

Then $0 < |x-3| < \delta \Rightarrow$

$2 < x < 4$

$0 < x-2 < 2$

$\Rightarrow |x-2| < 2$

Proof Let $\varepsilon > 0$. Define $\delta = \min\{1, \frac{\varepsilon}{2}\}$

Then if $0 < |x-3| < \delta$, then $|f(x) - L| = |x^2-5x+4-(-2)|$

$|x^2-5x+6| = |x-2||x-3| < 2 \cdot \delta \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon$ ■