$$\int \csc^6(x)\cot(x)\,dx = -\int \overset{u^5}{\csc^5(x)}\,\overset{\frac{du}{dx}\cdot dx}{(-\csc(x)\cot(x))dx}$$

$u = \csc(x)$

$du = -\csc(x)\cot(x)\,dx$

$$= -\int u^5\,du = -\frac{u^6}{6} + C = -\frac{\csc^6(x)}{6} + C$$

$$\int \sin^6(x)\cos^3(x)\,dx$$

$$= \int \sin^6(x)\left(\cos^2(x)\right)\cos(x)\,dx$$

$$= \int \sin^6(x)(1-\sin^2(x))\cos(x)\,dx$$

$$= \int \sin^6(x)\cos(x)\,dx - \int \sin^8(x)\cos(x)\,dx$$

$$= \frac{\sin^7(x)}{7} - \frac{\sin^9(x)}{9} + C$$

$$\int \sin^2(x)\,dx = \int \frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\int (1-\cos(2x))\,dx$$

$$= \frac{1}{2}\left[\int dx - \int \cos(2x)\,dx\right]$$

$$= \frac{1}{2}\left[\int dx - \frac{1}{2}\int \cos(2x)\cdot 2\,dx\right]$$

$$= \frac{1}{2}x - \frac{1}{2}\left[\sin(2x)\right] + C$$

$\cos(2x) = 2\cos^2(x) - 1 =$

$\sin(2x) = 2\sin(x)\cos(x)$

$\to 2\cos^2(x) - 1 = 2(1-\sin^2(x)) - 1$

$= 2 - 2\sin^2(x) - 1$

$= 1 - 2\sin^2(x) = \cos(2x)$

$\to -2\sin^2(x) = \cos(2x) - 1$

$\sin^2(x) = \dfrac{\cos(2x)-1}{-2}$

$= \dfrac{1-\cos(2x)}{2}$

Finish Trig antiderivatives

$\int \sin(x)\,dx = -\cos(x) + C$

$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

$\int \cos(x)\,dx = \sin(x) + C$

$\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$

$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}dx = -\int \frac{-\sin(x)\,dx}{\cos(x)} = -\int \frac{du}{u} = -\ln|u| + C$

$= -\ln|\cos(x)| + C = \ln\left(|\cos(x)|^{-1}\right) + C$

$= \ln\left(\frac{1}{|\cos(x)|}\right) + C = \ln\left|\frac{1}{\cos(x)}\right| + C$

$$\boxed{\int \tan(x)\,dx = \ln|\sec(x)| + C}$$

$\int \cot(x)\,dx = \int \frac{\cos(x)\,dx}{\sin(x)} = \int \frac{du}{u} = \ln|\sin(x)| + C$

$\int \sec(x)\,dx = \int \sec(x)\left(\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\right)dx = \int \left(\frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x)+\tan(x)}\right)dx$

Let $u = \sec(x) + \tan(x) \longrightarrow$

$du = \sec(x)\tan(x) + \sec^2(x)$

$= \int \frac{du}{u} = \ln|u| + C = \boxed{\ln|\sec(x) + \tan(x)| + C = \int \sec(x)\,dx}$

$$\int \csc(x)\,dx = \int \csc(x)\left(\frac{-\csc(x)-\cot(x)}{-\csc(x)-\cot(x)}\right)dx$$

$$= \int \frac{-\csc^2(x)-\csc(x)\cot(x)}{-\csc(x)-\cot(x)} = -\int \frac{-\csc^2(x)-\csc(x)\cot(x)}{\csc(x)+\cot(x)}\,dx$$

$$\left(\begin{array}{l} u=\csc(x)+\cot(x) \longrightarrow \\ du=\left(-\csc(x)\cot(x)-\csc^2(x)\right)dx \end{array}\right)$$

$$= -\int \frac{du}{u} = -\ln|u|+C = -\ln|\csc(x)+\cot(x)|+C$$

$$\text{or } \ln\left|\frac{1}{\csc(x)+\cot(x)}\right|+C$$

$$\text{or } \ln\left|\frac{\csc(x)-\cot(x)}{\csc^2(x)-\cot^2(x)}\right|+C$$

$$\text{or } \ln\left|\frac{\csc(x)-\cot(x)}{\cot^2(x)+1-\cot^2(x)}\right|+C = \boxed{\begin{array}{l}\ln|\csc(x)-\cot(x)|+C \\ = \int \csc(x)\,dx\end{array}}$$

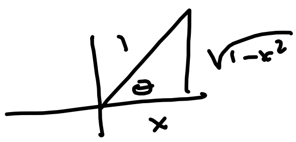
Bonus: Derivatives of Inverse Trig Functions:

Recall $(g^{-1})'(x) = \dfrac{1}{g'(g^{-1}(x))}$

$g(x) = \cos(x) \longrightarrow g'(x) = -\sin(x)$

$(g^{-1})(x) = \arccos(x) \longrightarrow$

$(g^{-1})'(x) = \dfrac{1}{-\sin(\arccos(x))} = \dfrac{1}{-\sqrt{1-x^2}}$



$\dfrac{d}{dx}\left[\arccos(x)\right] = -\dfrac{1}{\sqrt{1-x^2}}$

You can do something similar for $\dfrac{d}{dx}\left[\arcsin(x)\right]$

Integration by Parts BONUS MATERIAL

When you can't integrate, but you can differentiate.

$$(uv)' = u'v + uv'$$

$$\frac{d(uv)}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

$$\int \frac{d(uv)}{dx}\,dx = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx$$

$$uv = \int v\,du + \int u\,dv = uv$$

$$\Rightarrow \boxed{\int u\,dv = uv - \int v\,du}$$

EXAMPLE

$$\int x\sin(x)\,dx$$

$u = x$ $\quad dv = \sin(x)\,dx$

$du = dx$ $\quad v = -\cos(x)$

$$\int u\,dv = uv - \int v\,du = x(-\cos(x)) - \int(-\cos(x))\,dx$$

$$= \boxed{-x\cos(x) + \sin(x) + C = \int x\sin(x)\,dx} = F(x) + C$$

$$\Rightarrow F'(x) = -\cos(x) + x\sin(x) + \cos(x) = x\sin(x) \checkmark$$

$$\int \underbrace{\ln(x)}_{u}\,\underbrace{dx}_{dv}$$

$u = \ln(x)$ $\quad dv = dx$

$du = \frac{1}{x}\,dx$ $\quad v = x$

$$\int u\,dv = uv - \int v\,du$$

$$= \ln(x)\cdot x - \int x\cdot\frac{1}{x}\,dx$$

$$= x\ln(x) - \int dx = \boxed{x\ln(x) - x + C = \int \ln(x)\,dx}$$

BIG APPLICATION

$\int \underbrace{\arccos(x)}_{u}\,\underbrace{dx}_{dv}$

Don't know, but we DO know

$\frac{d}{dx}[\arccos(x)] = -\frac{1}{\sqrt{1-x^2}}$, so, ...

Let $u = \arccos(x)$ $\quad dv = dx$

$du = -\frac{1}{\sqrt{1-x^2}}dx$ $\quad v = x$

$\Rightarrow \int u\,dv = uv - \int v\,du = (\arccos(x))(x) - \int x\left(-\frac{1}{\sqrt{1-x^2}}\right)dx$

$= x\arccos(x) + \int x(1-x^2)^{-\frac{1}{2}}dx$

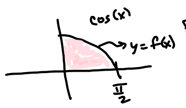
$u = 1-x^2$

$du = -2x\,dx$

$= x\arccos(x) - \frac{1}{2}\int \underbrace{(1-x^2)^{-\frac{1}{2}}}_{u^{-1/2}}\underbrace{(-2x\,dx)}_{du}$

$= x\arccos(x) - \frac{1}{2}(1-x^2)^{\frac{1}{2}}\cdot(2) + C$

$= x\arccos(x) - \sqrt{1-x^2} + C$



Find area of ↑ region in 2 ways:

$\int_a^b \sim dx$

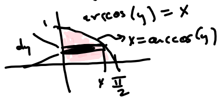
$\int_c^d \sim dy$

$\int_0^{\frac{\pi}{2}} \cos(x)\,dx = \sin(x)\Big]_0^{\frac{\pi}{2}}$

$= \sin(\frac{\pi}{2}) - \sin(0) = \boxed{1}$

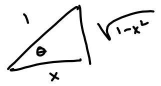
$y=\cos(x) \Rightarrow$

$\arccos(y) = x$



Area $= \int_0^1 \arccos(y)\,dy$

$\frac{d}{dx}[\arccos(x)] = \frac{1}{-\sin(\arccos(x))} = \frac{1}{-\sqrt{1-x^2}}$, etc.



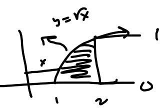
$= \left[x\arccos(x) - \sqrt{1-x^2}\right]_0^1$

$= 1\arccos(1) - \sqrt{1-1^2} - (0\arccos(0) - \sqrt{1-0})$

$= 0 - 0 - 0 - (-1) = 1$

$\int_1^2 \sqrt{x-1}\,dx = \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Big]_{1=x}^{2=x} = \frac{2}{3}\left[(x-1)^{\frac{3}{2}}\right]_1^2$

$= \frac{2}{3}\left[1^{3/2} - 0^{\frac{3}{2}}\right] = \frac{2}{3}$



$y = \sqrt{x-1}$

$y^2 = x-1$

$g(y) = y^2+1 = x$ $\quad f^{-1}(x) = x^2+1$

$\int_0^1 (\text{Right} - \text{left})\,dy = \int_0^1 (2 - (y^2+1))\,dy$

$= \int_0^1 (1-y^2)\,dy = \left[y - \frac{y^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$

Dingbat!

$$\int \ln(x)\,dx$$

$u = \ln(x)$ $\qquad dv = dx$

$du = \frac{1}{x}\,dx$ $\qquad v = x$

$$\int u\,dv = uv - \int v\,du = x\ln(x) - \int x\left(\frac{1}{x}\right)dx$$

$$= x\ln(x) - \int dx = x\ln(x) - x$$