§4.5 Substitution Rule (Chain Rule)

$$(-\sin(5x))(5) = \frac{d}{dx}(\,?\,)$$

$$= \frac{d}{dx}\left[\cos(5x)\right]$$

$$\longrightarrow \int -5\sin(5x)\,dx = \cos(5x)+C$$

$$\frac{d}{dx}\left[\sqrt{x^3+4x^2}\right] = \frac{d}{dx}\left[(x^3+4x^2)^{\frac{1}{2}}\right]$$

$$= \tfrac{1}{2}(x^3+4x^2)^{-\frac{1}{2}}(3x^2+8x)$$

$$\int \frac{3x^2+8x}{(x^3+4x^2)^{\frac{1}{2}}}\,dx = \int (x^3+4x^2)^{-\frac{1}{2}}(3x^2+8x)\,dx$$

$$= \int u^{-\frac{1}{2}}\frac{du}{dx}\,dx = \int u^{-\frac{1}{2}}\,du$$

$$= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = 2u^{\frac{1}{2}} + C$$

See the integrand as the derivative of something

$$\int \sin(5x)\,dx = \frac{1}{5}\int 5\sin(5x)\,dx$$

$$= \frac{1}{5}\int \sin(u)\,du = \frac{1}{5}\left(-\cos(u)\right) + C$$

$$= -\frac{1}{5}\cos(5x) + C$$

Recall Chain Rule

$$\frac{d}{dx}\left[f(u(x))\right] = \frac{df}{du}\cdot\frac{du}{dx} = f'(u)\,u'$$

→ Differential of f is

$\frac{df}{dx}\cdot dx$ (Multiply $\frac{df}{dx}$ by dx to get df

$$\int f(u(x))\,\underbrace{u'(x)\,dx}_{du} = \int f(u)\,du = F(u) + C,$$

where $F'(x) = f(x)$

$$\int (x^2+7)^{\frac{5}{3}}(x)\,dx = \frac{1}{2}\int (x^2+7)^{\frac{5}{3}}(2x\,dx) = \frac{1}{2}\int u^{\frac{5}{3}}\,du$$

$$= \frac{1}{2}\cdot\frac{3}{8}u^{\frac{8}{3}} + C$$

$$= \frac{3}{16}(x^2+7)^{\frac{5}{3}} + C$$

Let $u = x^2+7$

$\longrightarrow du = 2x\,dx$

STUDENTS LIKE THIS:

$u = x^2+7 \Longrightarrow$

$du = 2x\,dx \Longrightarrow$

$dx = \frac{du}{2x}$ This gives

$$\int (x^2+7)^{\frac{5}{3}}(x)\,dx = \int u^{\frac{5}{3}}(x)\left(\frac{du}{2x}\right)$$

$$= \int u^{\frac{5}{3}}\cdot\frac{du}{2} = \frac{1}{2}\int u^{\frac{5}{3}}\,du = \frac{1}{2}\left[\frac{u^{\frac{8}{3}}}{\frac{8}{3}}\right] + C$$

$$= \frac{1}{2}\left(\frac{3}{8}\left(x^2+7\right)^{\frac{8}{3}}\right) + C$$

$$= \frac{3}{16}(x^2+7)^{\frac{8}{3}} + C$$

$$\frac{1}{2}\left[\frac{u^{\frac{8}{3}}}{\frac{8}{3}} + C\right] = \frac{3}{16}u^{\frac{8}{3}} + \frac{1}{2}C = \frac{3}{16}(x^2+7)^{\frac{8}{3}} + \frac{1}{2}C$$

Let $\hat{C} = \frac{1}{2}C \longrightarrow \frac{3}{16}(x^2+7)^{\frac{8}{3}} + \hat{C}$

$$\int \sec^5(x)\tan(x)\,dx$$

$$\frac{d}{dx}\left[\sec(x)\right] = \sec(x)\tan(x)$$

$$d\left[\sec(x)\right] = \sec(x)\tan(x)\,dx$$

$$= \int \sec^4(x)\left(\sec(x)\tan(x)\,dx\right.$$

$$= \int u^4\,du, \text{ where } u = \sec(x)$$

Try: $u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$$dx = \frac{du}{\sec(x)\tan(x)}$$

$$\Longrightarrow \int \sec^5(x)\tan(x)\,dx = \int \sec^{\cancel{5}\,4}(x)\cancel{\tan(x)}\cdot\frac{du}{\cancel{\sec(x)\tan(x)}}$$

$$= \int \sec^4(x)\,du = \int u^4\,du = \frac{1}{5}u^5 + C = \boxed{\frac{1}{5}\sec^5(x) + C}$$

$$\int_0^{10} \frac{dx}{\sqrt{4x+9}}$$

$$\int_a^b f(\underbrace{g(x)}_{u})\underbrace{g'(x)\,dx}_{du} = \begin{cases} \int_{g(a)}^{g(b)} f(u)\,du \\ \int_{x=a}^{x=b} f(u)\,du \end{cases}$$

to avoid messing with $g(a)$ & $g(b)$

$$\int_0^{10} \frac{dx}{\sqrt{4x+9}} =$$

$$\left(\begin{array}{l} u = 4x+9 \\ du = 4\,dx \end{array} \right)$$

"My way"

$$= \frac{1}{4}\int_0^{10} \frac{4\,dx}{\sqrt{4x+9}} = \frac{1}{4}\int_{x=0}^{x=10} u^{-\frac{1}{2}}\,du = \frac{1}{4}\cdot 2u^{\frac{1}{2}}\Big]_{x=0}^{x=10}$$

$$= \frac{1}{2}(4x+9)^{\frac{1}{2}}\Big]_0^{10} = \frac{1}{2}\left[(4(10)+9)^{\frac{1}{2}} - (4(0)+9)^{\frac{1}{2}}\right]$$

$$= \frac{1}{2}\left[(49)^{\frac{1}{2}} - (9)^{\frac{1}{2}}\right] = \frac{1}{2}[7-3] = \frac{1}{2}(4) = \boxed{2}$$

more formally:

(Book Way)

$$\int_0^{10} \frac{dx}{\sqrt{4x+9}} =$$

$$\left(\begin{array}{ll} u = 4x+9 & u(0) = 9 \\ du = 4\,dx & u(10) = 49 \end{array} \right) \rightarrow \frac{du}{4} = dx$$

$$= \int_9^{49} \frac{\frac{du}{4}}{\sqrt{u}} = \int_9^{49} u^{-\frac{1}{2}}\,\frac{du}{4} = \frac{1}{4}\int_9^{49} u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4}\cdot 2u^{\frac{1}{2}}\Big]_9^{49} = \frac{1}{2}\left[49^{\frac{1}{2}} - 9^{\frac{1}{2}}\right] = \frac{1}{2}[7-3] = \frac{1}{2}[4] = \boxed{2}$$

New!

$$\int_0^{10} \frac{x\,dx}{\sqrt{4x+9}}$$

$$\left(\begin{array}{ll} u = 4x+9 & u(0) = 9 \\ du = 4\,dx & u(10) = 49 \\ dx = \frac{du}{4} & \end{array}\right)$$

$$= \int_9^{49} \frac{x\left(\frac{du}{4}\right)}{u^{\frac{1}{2}}}$$

DOH!

here's the trick:

$u = 4x + 9 = u$

$4x = u - 9$

$x = \frac{u-9}{4}$

$$= \int_9^{49} \frac{\left(\frac{u-9}{4}\right)\left(\frac{du}{4}\right)}{u^{\frac{1}{2}}} = \frac{1}{16}\int_9^{49} \frac{(u-9)}{u^{\frac{1}{2}}}\,du = \frac{1}{16}\int_9^{49} (u-9)u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{16}\int_9^{49} \left(u^{\frac{1}{2}} - 9u^{-\frac{1}{2}}\right)du = \frac{1}{16}\left[\frac{2}{3}u^{\frac{3}{2}} - 9\cdot 2u^{\frac{1}{2}}\right]_9^{49}$$

$$= \frac{1}{16}\left[\left(\frac{2}{3}(49)^{\frac{3}{2}} - 18(49)^{\frac{1}{2}}\right) - \left(\frac{2}{3}(9)^{\frac{3}{2}} - 18(9)^{\frac{1}{2}}\right)\right]$$

$$= \frac{1}{16}\left[\left(\frac{2}{3}(343) - 18(7)\right) - \left(\frac{2}{3}(27) - 18(3)\right)\right]$$

$$= \frac{1}{16}\left[\left(\frac{686}{3} - 126\right) - (18 - 54)\right]$$

$$= \frac{1}{16}\left[\frac{686}{3} - 126 + 36\right] = \frac{1}{16}\left[\frac{686}{3} - \frac{90}{1}\cdot\frac{3}{3}\right]$$

$$= \frac{1}{16}\left[\frac{686 - 270}{3}\right] = \frac{1}{16}\left[\frac{416}{3}\right] = \frac{416}{48} = \frac{208}{24} = \frac{104}{12} = \frac{52}{6}$$

$$= \boxed{\frac{26}{3}} = 8.\overline{6}$$

≈ 8.7 OR 8.67 OR 8.667

$$\left(49^{\frac{3}{2}}\right) = \left(49^{3}\right)^{\frac{1}{2}} = \left((49)^{\frac{1}{2}}\right)^{3} = 7^{3} = 343$$

$$\begin{array}{r} {}^{6}49 \\ \times\ 7 \\ \hline 343 \end{array}$$

$$\int_0^{10} \frac{x}{\sqrt{4x+9}}\,dx$$

$= 8.66666666667$