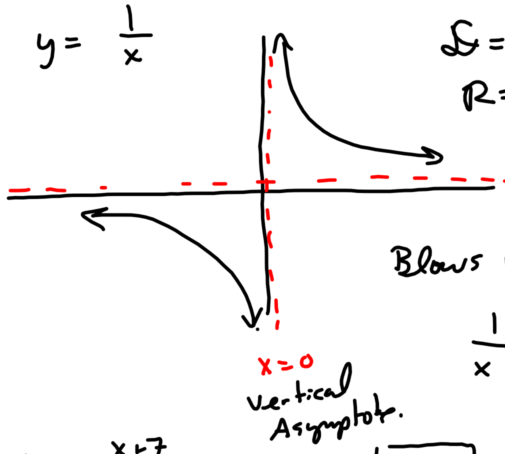
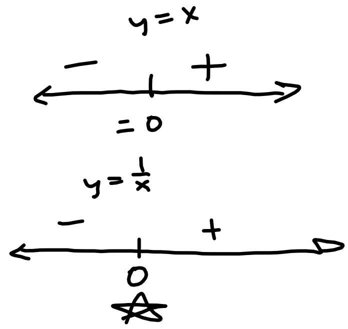
Rational Functions, End Behavior, Vertical Asymptotes

$y = \frac{1}{x}$



$\mathcal{D} = \mathbb{R} \setminus \{0\}$

$\mathcal{R} = \mathbb{R} \setminus \{0\}$



Blows up @ $x = 0$

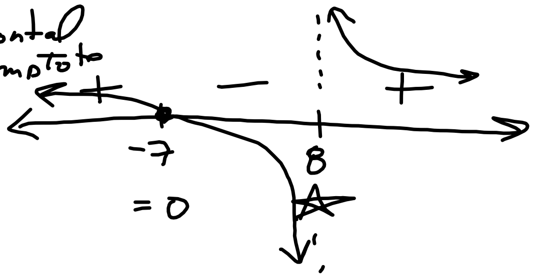
$\frac{1}{x} \xrightarrow{x \to \infty} 0$

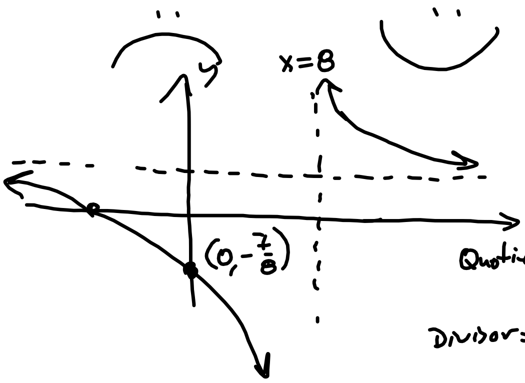
$\frac{1}{2}, \frac{1}{4}, \frac{1}{12}, \frac{1}{\text{Bezillion}} \approx 0$

$R(x) = \frac{x+7}{x-8} \xrightarrow{x \to \infty} \frac{x}{x} = \boxed{1 = y}$ Horizontal Asymptote

$x = 8$ V.A.

$(-7, 0) = x\text{-int}$





What's calculus say?

Quotient → $1 \text{ r } 15$ = remainder

Divisor $= x - 8 \overline{)\, x + 7}$ = Dividend

$\underline{-(x-8)}$

15

This says

$R(x) = \frac{x+7}{x-8} = 1 + \frac{15}{x-8} = 1 + 15(x-8)^{-1} \longrightarrow$

$R'(x) = -15(x-8)^{-2}$

$= \frac{-15}{(x-8)^2}$

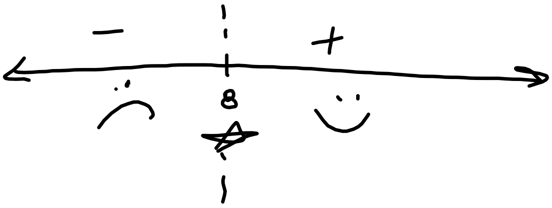
$x + 7 = (x-8)(1) + 15$

$R(x) = \frac{x+7}{x-8} = \frac{u}{v} \longrightarrow f'(x) = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$= \frac{1(x-8) - (x+7)(1)}{(x-8)^2} = \frac{x-8-x-7}{(x-8)^2} = \frac{-15}{(x-8)^2} = R'(x)$

$R''(x) = \frac{d}{dx}\left[-15(x-8)^{-2}\right] = 15(x-8)^{-3} = \frac{15}{(x-8)^3}$

$15 \to 30$ $15 \to 30$



Click on Link:

Rational Functions for College Algebra Students

This should really help review for rational functions for THIS class. Scroll down to #s 3, 4, and 5.

This class does some of the best work of any Calc I group I've had.

3.3, then rational functions, applications,
Newton's Method and Antiderivatives

It's all about zeros upstairs $(=0)$ & downstairs $(\bigstar)$

$$R(x) = \frac{3x^2+6x-24}{4x^2+27x+18} = \frac{(3x-6)(x+4)}{(x+6)(4x+3)} = \frac{3(x-2)(x+4)}{(x+6)(4x+3)}$$

Nothing cancels so No Holes

$-6, -4, -\frac{3}{4}, 2$

$\mathcal{D} = \mathbb{R} \setminus \{-6, -\frac{3}{4}\}$ $\boxed{x=-6,\ x=-\frac{3}{4} \text{ are vertical asymptotes}}$

$R=0 \Rightarrow x=-4, 2 \rightsquigarrow$ $\boxed{(-4,0),\ (2,0) \text{ are x-ints}}$

Limit @ infinity (END BEHAVIOR)

my way: $\frac{3x^2+\sim}{4x^2+\sim} \xrightarrow{|x|\to\infty} \frac{3x^2}{4x^2} = \boxed{\frac{3}{4} = y} = HA$. Nice when

deg(numer) = deg(denom)

$y = \frac{3}{4}$ = Horizontal Asymptote = ~~Hor. As.~~ = HA!

deg(numer) > deg(denom) Oblique Asymptotes $(ax+b,$ ax^2+bx+c

$\frac{6x^2-5x-7}{x+2}$ $2-1=1$ degree $y=ax+b$

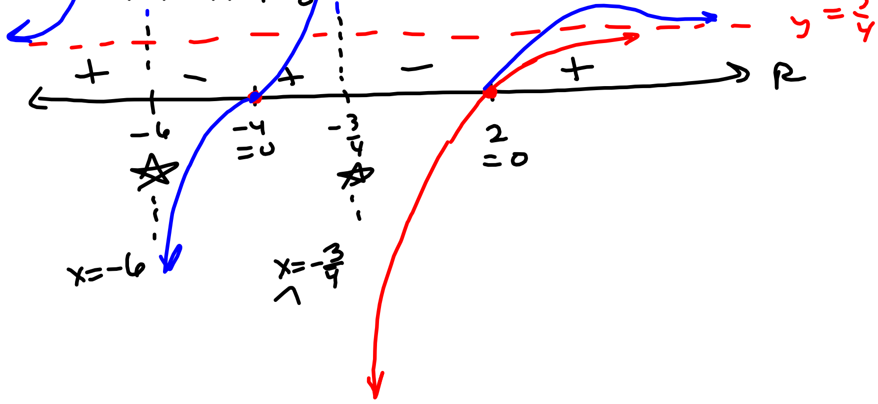
$\frac{6x^3+5x^2-7}{x+2}$ $3-1=2$ degree $y=ax^2+bx+c$

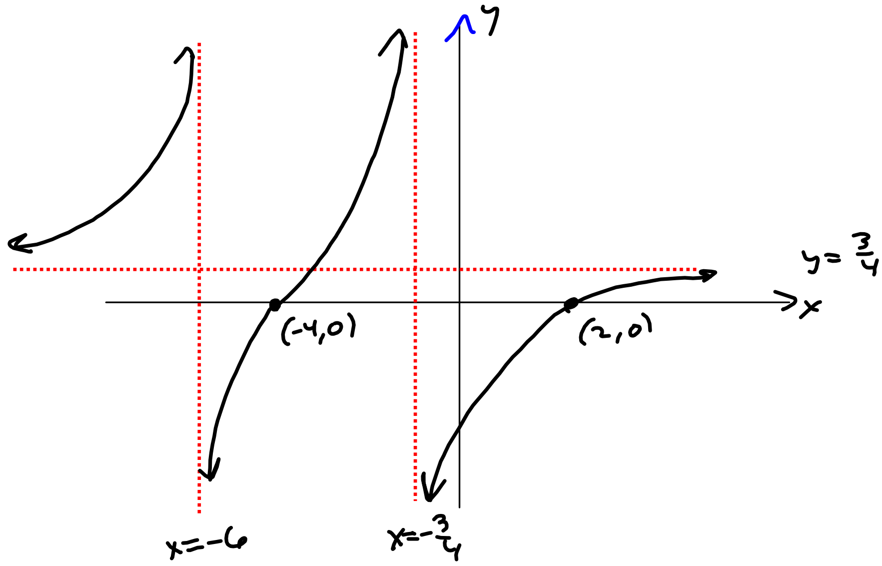
deg(numer) < deg(denom) ("Proper")

$\frac{7x}{x^2+2} \xrightarrow{x\to\pm\infty} 0$

$\frac{9x^{50}+7x^{40}-5x^{17}}{x^{51}+5} \xrightarrow{x\to\pm\infty} 0$

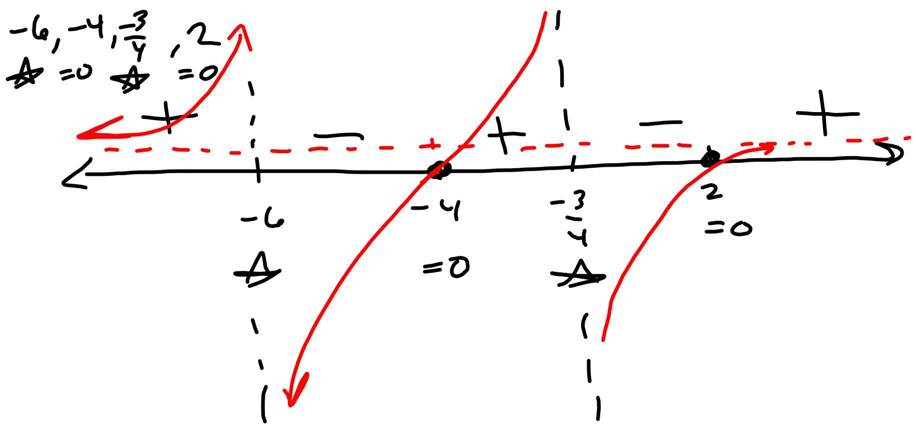
$$R(x) = \frac{3x^2+6x-24}{4x^2+27x+18} = \frac{(3x-6)(x+4)}{(x+6)(4x+3)}$$





$$\frac{3x^2+6x-24}{4x^2+27x+18} = \frac{\cancel{x^2}\left(3+\frac{6}{x}-\frac{24}{x^2}\right)}{\cancel{x^2}\left(4+\frac{27}{x}+\frac{18}{x^2}\right)} \xrightarrow{x \to \pm\infty} \frac{3}{4}$$

$x \longrightarrow \pm\infty$ is kind of like $|x| \longrightarrow \infty$



$$\frac{3x^2+6x-24}{4x^2+27x+18} = R(x) = \frac{u}{v} \longrightarrow R' = \left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$$

$$= \frac{(6x+6)(4x^2+27x+18)-(3x^2+6x-24)(8x+27)}{(4x^2+27x+18)^2}$$

This is as far as you'd go on a test question asking for R'' under a time control.

We were running out of time at the end of class, so we went to the technology crutch.

Continuing...

$$\frac{(6x+6)(4x^2+27x+18)-(3x^2+6x-24)(8x+27)}{(4x^2+27x+18)^2}$$

$$= \frac{6(x+1)(4x^2+27x+18)-(3x^2+6x-24)(8x+27)}{(4x^2+27x+18)^2}$$

$$\frac{6(4x^3+27x^2+18x+4x^2+27x+18)-(24x^3+81x^2+48x^2+162x-192x-648)}{(4x+3)^2(x+6)^2}$$

$$= \frac{24x^3+162x^2+108x+24x^2+162x+108-(24x^3+129x^2-30x-648)}{((4x+3)(x+6))^2}$$

$$= \frac{24x^3+186x^2+270x+108-24x^3-129x^2+30x+648}{(\quad)^2}$$

$$= \frac{57x^2+300x+756}{(\quad)^2} \quad \text{set } 0 \longrightarrow$$

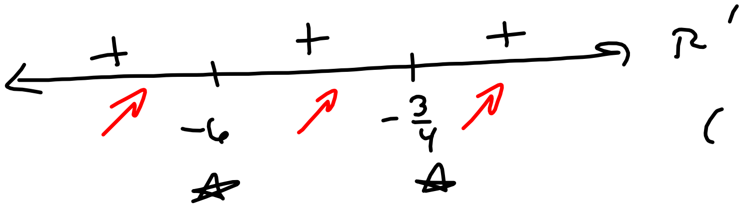
$ax^2+bx+c=0$, where $a=57$, $b=300$, $c=756$

$\longrightarrow b^2-4ac = 300^2-4(57)(756) = 90000-(228)(756)$

$= 90000-162334 < 0 \longrightarrow$ No real zeros.

Denom $= 0$ @ $-\frac{3}{4}, -6$

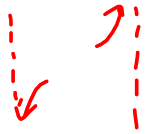
$$\begin{array}{r} 27 \\ \times 24 \\ \hline 108 \\ 540 \\ \hline 648 \end{array} \qquad \begin{array}{r} 756 \\ \times 228 \\ \hline 6014 \\ 15120 \\ 151200 \\ \hline 162334 \end{array}$$



$(\quad)^2$ doesn't change sign

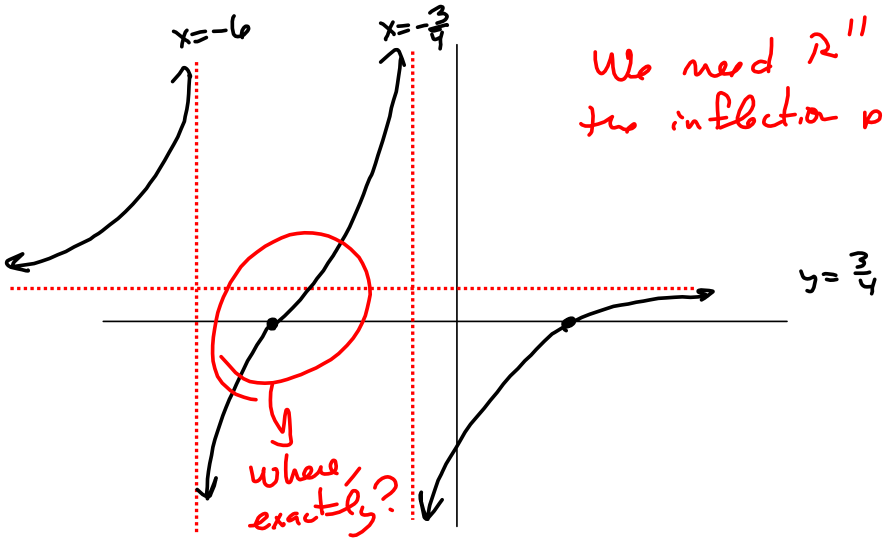
The sign pattern for R' is totally in agreement with the College Algebra.

What the College Algebra can't do is find the inflection point that must exist, because (look and think) is has to approach + infinity as x -> -3/4 from the left and it has to approach - infinity as it approaches x = - 6 from the right.



Somewhere in between it goes from ☹ to ☺

We need R'' to find the inflection point (mood change)



$$\frac{57x^2+300x+756}{\left((x+6)(4x+3)\right)^2} = R'(x)$$

$$R''(x) = \frac{(114x+300)\left((x+6)(4x+3)\right)^2}{\left(\left((x+6)(4x+3)\right)^2\right)^2}$$

$$- \frac{(57x^2+300x+756)\left(2\left((x+6)(4x+3)\right)^1(8x+27)\right)}{\left(\left((x+6)(4x+3)\right)^2\right)^2}$$

$$= \frac{(x+6)(4x+3)\left((114x+300)(x+6)(4x+3) - (57x^2+300x+756)(8x+27)\right)}{\left((x+6)(4x+3)\right)^4}$$

$$= \frac{(114x+300)(4x^2+27x+18) - (8x+27)(57x^2+300x+756)}{\left((x+6)(4x+3)\right)^3}$$

$$\begin{array}{r} {}^{4}\,57 \\ 27 \\ \hline {}^{1}\,399 \\ 1140 \\ \hline 1539 \end{array}$$

$$= \frac{456x^3+3078x^2+2052x+1200x^2+8100x+5400}{(\quad)^3}$$

$$+ \frac{-456x^3-2400x^2-6048x-1539x^2-8100x-20412}{(\quad)^3}$$

$$\begin{array}{r} {}^{4}\,{}^{4}\,756 \\ 8 \\ \hline 6048 \end{array}$$

$$= \frac{4278x^2+10152x+5400-3939x^2-14148x-20412}{(\quad)^3}$$

$$\begin{array}{r} {}^{5}\,57 \\ 8 \\ \hline 456 \end{array} \qquad \begin{array}{r} {}^{1}\,{}^{3}\,114 \\ 18 \\ \hline 912 \\ 1140 \\ \hline 2052 \end{array}$$

$$= \frac{339x^2+6213x-15012}{(x+6)^3(4x+3)^3}$$

$$\begin{array}{r} {}^{1}\,{}^{4}\,756 \\ 27 \\ \hline 5292 \\ 15120 \\ \hline 20412 \end{array} \qquad \begin{array}{r} {}^{2}\,114 \\ 27 \\ \hline 798 \\ 2280 \\ \hline 3078 \end{array}$$

$R''(x) = \ldots$

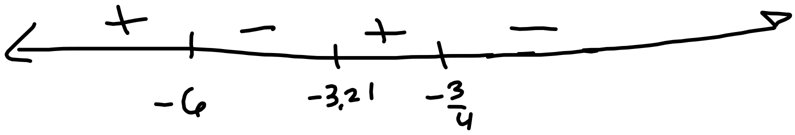
$R''(x) = 0 \Rightarrow x \approx -3.121026351$

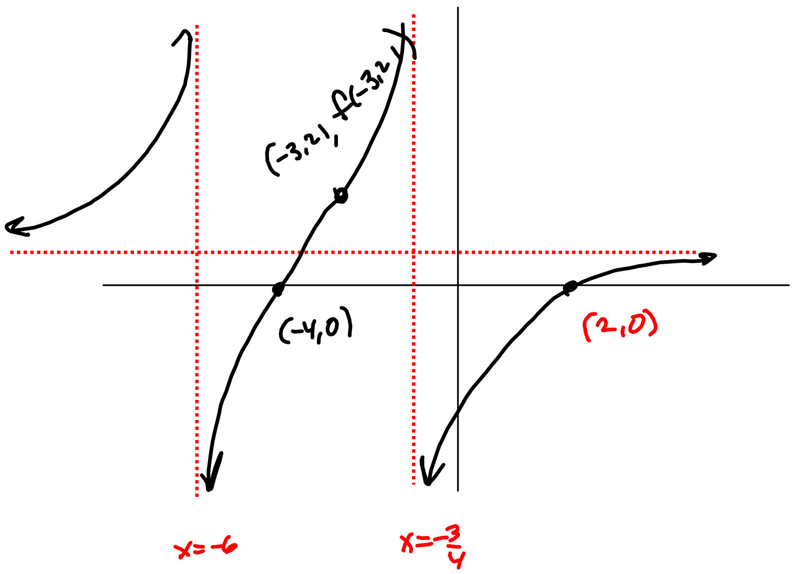
& Remember the denominator has same zeros as R & R'. Just check the multiplicities of them to know if you have a sign change.

-6 & $-\frac{3}{4}$ are the zeros

$$R''(x) = \frac{-456x^3 - 3600x^2 - 18144x - 35424}{(x-6)^3(4x+3)^3}$$ SET $= 0 \Rightarrow$

$x \approx -3.21,$
$x \approx -2.39$





What can be on midterm for graphing?

A "nice" cubic polynomial

A "nice" Rational Function

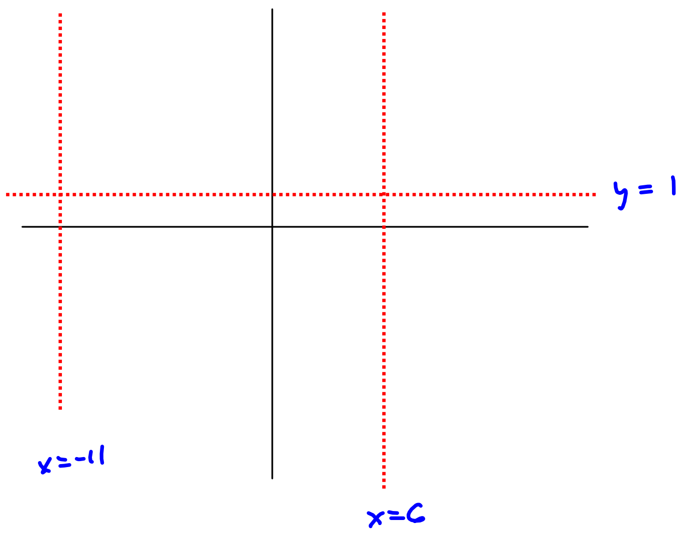
$$\frac{x+5}{x-6}$$

Find all asymptotes for a rational function

$$R(x) = \frac{(x+2)(x+3)}{(x-6)(x+11)} = \frac{x^2+5x+6}{x^2+5x-66}$$

V.A.: $x=6,\ x=-11$

H.A.: $y=1$



$$R(x) = \frac{x^3+2x^2-3x+5}{x^2-5x+2}$$

OBLIQUE / Slant Asymptote by Long Division!

$$\begin{array}{r l}
 & x+7 \\
x^2-5x+2 & \overline{)\,x^3+2x^2-3x+5} \quad \leftarrow \text{Dividend} \\
 & -(x^3-5x^2+2x) \\
\hline
 & 7x^2-5x
\end{array}$$

Divisor

$y = x+7$ is slant asymptote

To convince you:

$$\begin{array}{r l}
 & x+7 \;\; r\; 30x-9 \\
x^2-5x+2 & \overline{)\,x^3+2x^2-3x+5} \quad \leftarrow \text{Dividend} \\
 & -(x^3-5x^2+2x) \\
\hline
 & 7x^2-5x+5 \\
 & -(7x^2-35x+14) \\
\hline
 & 30x-9
\end{array}$$

Divisor

This says $\boxed{R(x) = x+7 + \frac{30x-9}{x^2-5x+2}}$

(Usually we report these as

$x^3+2x^2-3x+5 = (x+7)(x^2-5x+2) + 30x-9$

Dividend $=$ (quotient)(divisor) $+$ remainder)

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$ is what we're doing

Midterm Review

$$\lim_{x\to 2} \frac{x^2-4}{x^3-8} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)} = \lim_{x\to 2} \frac{x+2}{x^2+2x+4}$$

$$= \frac{4}{4+4+4} = \frac{4}{12} = \boxed{\frac{1}{3}}$$

$$\lim_{x\to 2} \frac{x^2-4}{x^3-8} = \lim f$$

$$f(x) = \frac{x^2-4}{x^3-8} = \frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)} = \underset{(x\neq 2)}{\frac{x+2}{x^2+2x+4}} \xrightarrow{x\to 2} \cdots \boxed{\frac{1}{3} = \lim f}$$

Find $\lim\limits_{x\to -7} \dfrac{3x^2+17x-28}{4x^2+31x+21} = \lim f$

$$\frac{3x^2+17x-28}{4x^2+31x+21} = \frac{3x^2+21x-4x-28}{4x^2+28x+3x+21}$$

$3\cdot 2\cdot 2\cdot 7$

$2\cdot 2\cdot 3\cdot 7$

$$= \frac{3x(x+7)-4(x+7)}{4x(x+7)+3(x+7)} = \frac{(x+7)(3x-4)}{(x+7)(4x+3)} = \frac{3x-4}{4x+3} \xrightarrow{x\to -7} \frac{-21-4}{-25} =$$

$x \neq -7$

$$= \boxed{1 = \lim f}$$

$\lim\limits_{x\to -7} \dfrac{3x^2-25x+28}{(4x+3)(x+7)} = \lim f$

$$\frac{3x^2-25x+28}{(4x+3)(x+7)} = \frac{3x^2-21x-4x+28}{\text{same}} = \frac{3x(x-7)-4(x-7)}{\text{same}}$$

$$= \frac{(x-7)(3x-4)}{(x+7)(4x+3)} \xrightarrow{x\to -7} \nexists.$$

$$\boxed{\lim f \nexists.}$$

$\lim\limits_{x\to 5} f(x) = \lim\limits_{x\to 5} (7x-3) = 32 = L$

proof Let $\varepsilon > 0$ be given

Want $|f(x)-L| < \varepsilon$ whenever $0 < |x-5| < \delta$

$\Longleftrightarrow |7x-3-32| = |7x-35| = 7|x-5| < 7\delta = \varepsilon$

$\Longleftrightarrow \delta = \frac{\varepsilon}{7}$.

Alternate (my way)

Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{7} = \frac{\varepsilon}{\text{slope}}$: Then

$0 < |x-5| < \delta \Longrightarrow |f(x)-L| = |7x-3-32| = 7|x-5|$

$< 7\delta = 7\left(\frac{\varepsilon}{7}\right) = \varepsilon$ ■