Week 8 Assignment is Posted. Due 3/24 (after Spring Break)

f is **increasing** on an interval *I* if $f(x_1) < f(x_2)$ whenever $x_1 < x_3$ in *I*.

• $(x_2, f(x_2))$

f is **decreasing** on an interval *I* if $f(x_1) > f(x_2)$ whenever $x_1 < x_3$ in *I*.

• $(x_1, f(x_1))$

These definitions give rise to *closed* intervals of increase/decrease, so there was an overlap between intervals of increase and decrease at local max/min points.

NOW, in Section 3.3, we're looking for the *interior* of those intervals, i.e., *open intervals* of increase/decrease. The book sort of finesses the whole thing by asking for *open* intervals of increase/decrease, eliminating the overlap of increase/decrease at local max/min points.

This is a better way of looking at it, since $f' > 0$ means f is increasing and $f' < 0$ means f is decreasing. $f' = 0$ or f' undefined are taken out of consideration. These are important (critical) points where max/min values might be found. They're the Boundary Points of intervals of increase/decrease, when they correspond to max/min points* on the graph.

$f'=0$ $\quad -x^{\frac{2}{3}}$ $\quad$ Cusp situation $\quad x^{\frac{2}{3}}$

***Remember that critical numbers are candidates for max/min but might not be, for instance terrace points and points where the function has a vertical tangent at a critical point, which I haven't shown you, yet.**

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Proof uses MVT. I'll prove (a), briefly.

(a) Suppose $f' > 0$ on (a,b) Then

Let $x_1 < x_2$ in (a,b). Then

$m_{AVG} = \dfrac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$ for some $c \in (a,b)$, by MVT.

$f'(c) > 0$, b/c $c \in (a,b)$

Also $x_2 - x_1 > 0$, b/c $x_1 < x_2$

So $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \longrightarrow$

$$f(x_2) - f(x_2) = \underbrace{f'(c)}_{>0}\underbrace{(x_2-x_1)}_{>0} > 0$$

$\Longrightarrow f(x_2) - f(x_1) > 0$

$\Longrightarrow f(x_2) > f(x_1)$, i.e.

$f(x_1) < f(x_2)$, i.e., f is increasing on (a,b) ■

The First Derivative Test Suppose that c is a critical number of a continuous function f.

↗ Max ↘

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c. Terrace Point

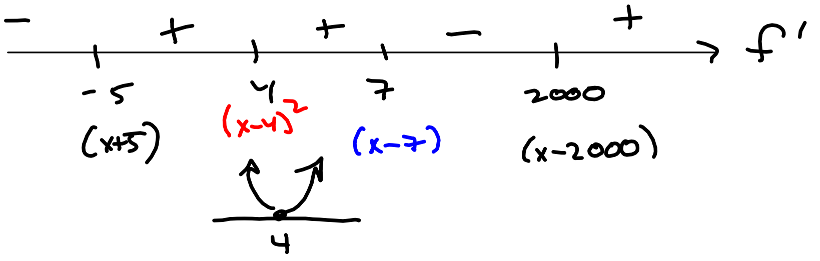
These are good words, and it's good to understand this, formally, but informally/semi-formally, it's very easy to understand and apply, if you can build and interpret a sign pattern. The arrows point the way. You're not slavishly referring to 3 bullet points when working these. You're just analyzing a sign pattern and understanding what it represents.

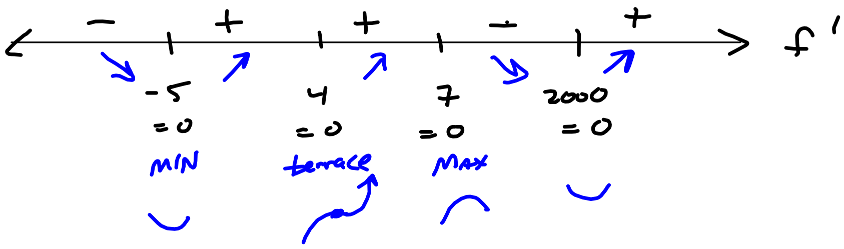
Polynomial-type situation. Suppose f' has the following sign pattern. Assume f is continuous on the entire real line.

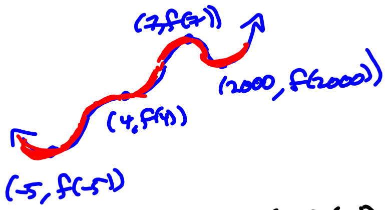
$f'(x) = (x+5)(x-4)^2(x-7)(x-2000)$ in factored form

$= x \cdot x^2 \cdot x \cdot x = x^5 +$ smaller stuff

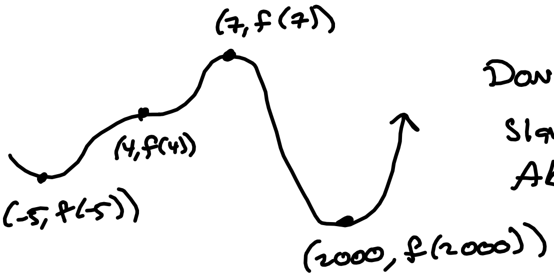
x^5 ↙ .... ↗ End Behavior for x^{odd}







Knowing no more details, either picture is just as legit.

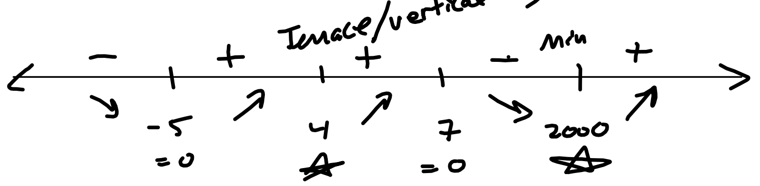


DON'T BE A slave to scale.
Abandon tickmarks in graphs.
Label key points.

Fractional-exponent situation: Assume that all the points are critical values.

"☆" means $x \notin \mathcal{D}(f')$ (zero in denominator, typically)

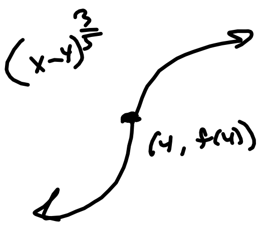
"critical" means $x \in \mathcal{D}(f)$, among other things.



$(x-4)^{\frac{3}{5}}$ likely involved

Almost does it

$$\frac{(x+5)(x-7)}{(x-4)^2(x-2000)}$$



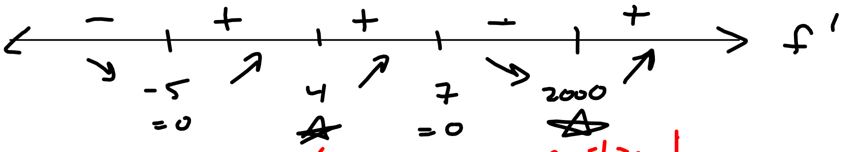
Goes vertical @ $x=4$

$f(4)=0$ $x=4 \in \mathcal{D}$, but

$f'(x) = \frac{3}{5}(x-4)^{-\frac{2}{5}}$

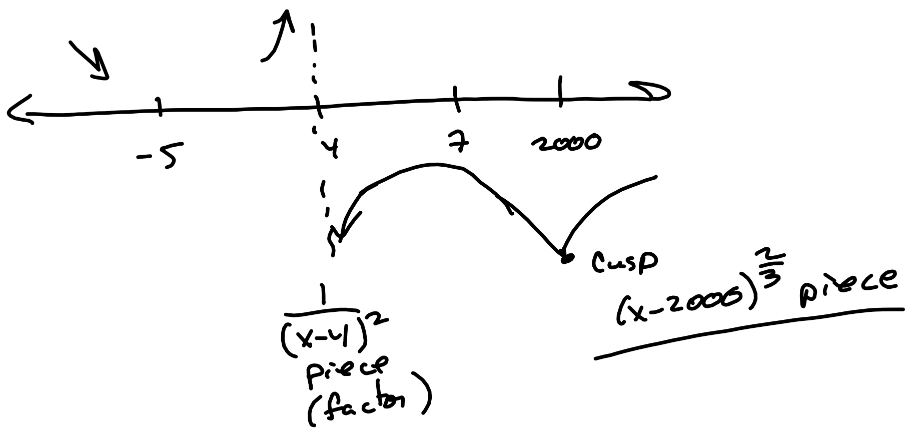
$4 \notin \mathcal{D}(f')$

This time, suppose that x = 2000 is in the domain of f, but x = 4 is not.



Not critical point, so probably a vertical asymptote for f.

critical point cusp



Sign Patterns:

Factored polynomials and rational functions are the best.

After that come trig functions that are easy to "see." Not all are.

Sign patterns for the 2 cases above are quick and easy.

All other cases? Plug in a test value in each subinterval. Evaluate just far enough to determine the sign. Learn just how much you have to turn the crank for the decision.

Product-to-sum

$\sin(u)\cos(v)$

Product-to-Sum Derivations

$\sin(u+v) = \sin u\cos v + \sin v\cos u$
$\sin(u-v) = \sin u\cos v - \sin v\cos u$

$\sin(u+v) + \sin(u-v) = 2\sin u\cos v \longrightarrow$

$$\boxed{\sin u\cos v = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]}$$

$\cos(u+v) = \cos u\cos v - \sin u\sin v$
$\cos(u-v) = \cos u\cos v + \sin u\sin v$

$\cos(u+v) + \cos(u-v) = 2\cos u\cos v \longrightarrow$

$$\boxed{\cos u\cos v = \frac{1}{2}\left[\cos(u+v) + \cos(u-v)\right]}$$

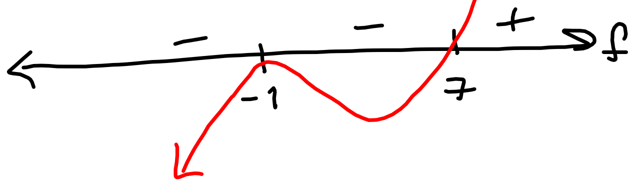
$-(\cos(u+v) = \cos u\cos v - \sin u\sin v)$
$+\ \cos(u-v) = \cos u\cos v + \sin u\sin v$

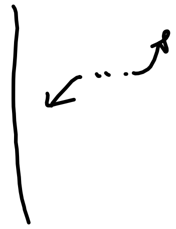
$\cos(u-v) - \cos(u+v) = 2\sin u\sin v \longrightarrow$

$$\boxed{\sin u\sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]}$$

Week 8 prep:

$f(x) = (x-7)^3(x+1)^2 = x^3 \cdot x^2 = x^5$





$f'(x) = 3(x-7)^2(x+1)^2 + (x-7)^3(2)(x+1)$

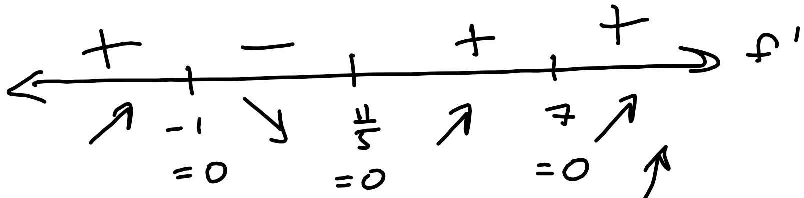
$= (x-7)^2(x+1)\left[3(x+1) + (x-7)(2)\right]$

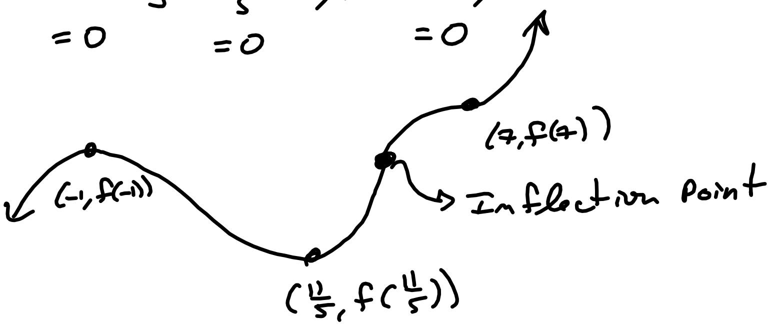
$= (x-7)^2(x+1)\left[3x+3+2x-14\right]$

$= (x-7)^2(x+1)\left[5x-11\right]^1 \overset{\text{SET}}{=} 0$

$x = 7, -1, \frac{11}{5} = -1, \frac{11}{5}, 7$

$x^2 x \cdot 5x = 5x^4$

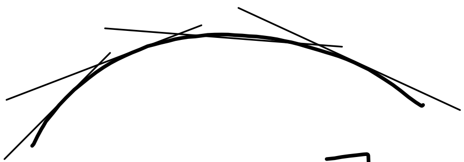




Concave up: f lives above all its tangents

$f'' > 0$

Concave down: f lives below all its tangents



$f'' < 0$

$$f''(x) = \frac{d}{dx}\left[(x-7)^2(x+1)\left[5x-11\right]\right] = 2(x-7)(x+1)(5x-11) + (x-7)^2(5x-11) + (x-7)^2(x+1)(5)$$

$$= 4(x-7)(5x^2-22x+5) \overset{\text{SET}}{=} 0$$

$x=7$ OR $5x^2-22x+5=0$

$$5\left(x^2-\frac{22}{5}x+1\right)=0$$

$$x^2-\frac{22}{5}x+\left(\frac{11}{5}\right)^2=-1+\frac{121}{25}=\frac{-25+121}{25}$$

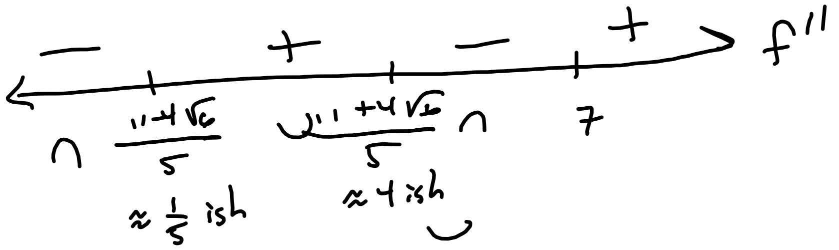
$$\left(x-\frac{11}{5}\right)^2=\frac{96}{25}$$

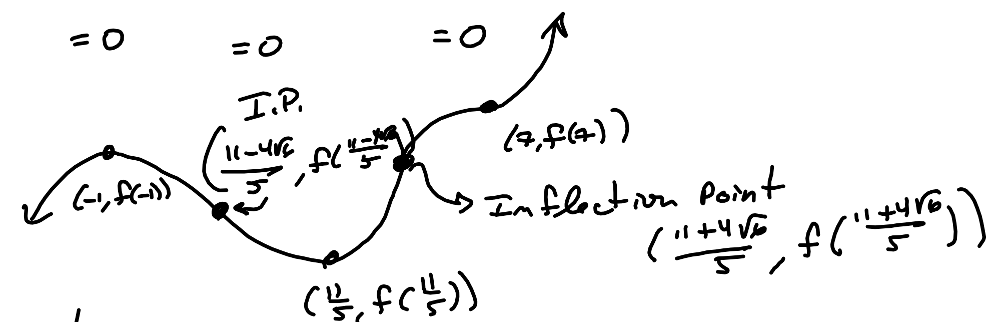
$$x=\frac{11\pm\sqrt{96}}{5}$$

$$=\frac{11\pm 4\sqrt{6}}{5}$$

2 | 96
2 | 48
2 | 24
2 | 12
2 | 6
3

$x=7, \frac{11\pm 4\sqrt{6}}{5}$ where $f''=0$





2nd Deriv. Test

$f'=0$, $f''>0$ Min

$f'=0$, $f''<0$ MAX

Usually 1st derivative test is all you need.